Announcement

Homework #4 has been posted on Blackboard and course website.

Due 2:50pm, Thursday, Oct 6
Today’s Agenda

• Filtering in Frequency domain
### 2D DFT Properties (Symmetry)

<table>
<thead>
<tr>
<th>Spatial Domain</th>
<th>Frequency Domain</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x, y)$ real</td>
<td>$F^*(u, v) = F(-u, -v)$</td>
<td>Even (symmetric): $w_e(x, y) = w_e(M - x, N - y)$</td>
</tr>
<tr>
<td>$f(x, y)$ imaginary</td>
<td>$F^*(-u, -v) = -F(u, v)$</td>
<td>Odd (antisymmetric): $w_o(x, y) = -w_o(M - x, N - y)$</td>
</tr>
<tr>
<td>$f(x, y)$ real</td>
<td>$R(u, v)$ even; $I(u, v)$ odd</td>
<td></td>
</tr>
<tr>
<td>$f(x, y)$ imaginary</td>
<td>$R(u, v)$ odd; $I(u, v)$ even</td>
<td></td>
</tr>
<tr>
<td>$f(-x, -y)$ real</td>
<td>$F^*(u, v)$ complex</td>
<td></td>
</tr>
<tr>
<td>$f(-x, -y)$ complex</td>
<td>$F(-u, -v)$ complex</td>
<td></td>
</tr>
<tr>
<td>$f^*(x, y)$ complex</td>
<td>$F^*(-u - v)$ complex</td>
<td></td>
</tr>
<tr>
<td>$f(x, y)$ real and even</td>
<td>$F(u, v)$ real and even</td>
<td></td>
</tr>
<tr>
<td>$f(x, y)$ real and odd</td>
<td>$F(u, v)$ imaginary and odd</td>
<td></td>
</tr>
<tr>
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<td>$f(x, y)$ complex and odd</td>
<td>$F(u, v)$ complex and odd</td>
<td></td>
</tr>
</tbody>
</table>

$\dagger$Recall that $x, y, u,$ and $v$ are discrete (integer) variables, with $x$ and $u$ in the range $[0, M - 1]$, and $y$, and $v$ in the range $[0, N - 1]$. To say that a complex function is even means that its real and imaginary parts are even, and similarly for an odd complex function.
### 2D DFT Properties (cont.)

<table>
<thead>
<tr>
<th>Name</th>
<th>DFT Pairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) Symmetry properties</td>
<td>See Table 4.1</td>
</tr>
<tr>
<td>2) Linearity</td>
<td>[ af_1(x, y) + bf_2(x, y) \Leftrightarrow aF_1(u, v) + bF_2(u, v) ]</td>
</tr>
</tbody>
</table>
| 3) Translation (general)                  | \[ f(x, y)e^{j2\pi(u_0x/M + v_0y/N)} \Leftrightarrow F(u - u_0, v - v_0) \]  
                                           | \[ f(x - x_0, y - y_0) \Leftrightarrow F(u, v)e^{-j2\pi(ux_0/M + vy_0/N)} \] |
| 4) Translation to center of the frequency | \[ f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2) \]  
   rectangle, \((M/2, N/2)\)       | \[ f(x - M/2, y - N/2) \Leftrightarrow F(u, v)(-1)^{u+v} \] |
| 5) Rotation                               | \[ f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0) \]  
                                           | \[ x = r \cos \theta \quad y = r \sin \theta \quad u = \omega \cos \varphi \quad v = \omega \sin \varphi \] |
| 6) Convolution theorem†                   | \[ f(x, y) \star h(x, y) \Leftrightarrow F(u, v)H(u, v) \]  
                                           | \[ f(x, y)h(x, y) \Leftrightarrow F(u, v) \star H(u, v) \] |

*Table 4.3*  
Summary of DFT pairs. The closed-form expressions in 12 and 13 are valid only for continuous variables. They can be used with discrete variables by sampling the closed-form, continuous expressions.

(Continued)
### 2D DFT Properties (cont.)

<table>
<thead>
<tr>
<th>Name</th>
<th>DFT Pairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>7) Correlation theorem†</td>
<td>$f(x, y) \ast h(x, y) \iff F^*(u, v)H(u, v)$</td>
</tr>
<tr>
<td></td>
<td>$f^*(x, y)h(x, y) \iff F(u, v) \ast H(u, v)$</td>
</tr>
<tr>
<td>8) Discrete unit impulse</td>
<td>$\delta(x, y) \iff 1$</td>
</tr>
<tr>
<td>9) Rectangle</td>
<td>$\text{rect}[a, b] \iff ab\frac{\sin(\pi ua)}{(\pi ua)} \frac{\sin(\pi vb)}{(\pi vb)} e^{-j\pi(ua+vb)}$</td>
</tr>
<tr>
<td>10) Sine</td>
<td>$\sin(2\pi u_0 x + 2\pi v_0 y) \iff$</td>
</tr>
<tr>
<td></td>
<td>$j\frac{1}{2} \left[ \delta(u + Mu_0, v + Nv_0) - \delta(u - Mu_0, v - Nv_0) \right]$</td>
</tr>
<tr>
<td>11) Cosine</td>
<td>$\cos(2\pi u_0 x + 2\pi v_0 y) \iff$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{2} \left[ \delta(u + Mu_0, v + Nv_0) + \delta(u - Mu_0, v - Nv_0) \right]$</td>
</tr>
</tbody>
</table>

The following Fourier transform pairs are derivable only for continuous variables, denoted as before by $t$ and $z$ for spatial variables and by $\mu$ and $\nu$ for frequency variables. These results can be used for DFT work by sampling the continuous forms.

12) Differentiation (The expressions on the right assume that $f(\pm\infty, \pm\infty) = 0$.)

\[
\left( \frac{\partial}{\partial t} \right)^m \left( \frac{\partial}{\partial z} \right)^n f(t, z) \iff (j2\pi \mu)^m(j2\pi \nu)^n F(\mu, \nu)
\]

13) Gaussian

\[
A2\pi\sigma^2 e^{-2\pi^2\sigma^2(t^2+z^2)} \iff Ae^{-(\mu^2+\nu^2)/2\sigma^2} \quad (A \text{ is a constant})
\]

†Assumes that the functions have been extended by zero padding. Convolution and correlation are associative, commutative, and distributive.
Basics of Filtering in Frequency Domain

**FIGURE 4.29** (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)
Frequency Domain Filtering Fundamentals

\[ g(x, y) \leftrightarrow H(u, v)F(u, v) \]

Centered \( F(M/2, N/2) = MN \bar{f}(x, y) \) \( \text{dc term} \)

FIGURE 4.30
Result of filtering the image in Fig. 4.29(a) by setting to 0 the term \( F(M/2, N/2) \) in the Fourier transform.
Low-Pass and High-Pass Filters

**FIGURE 4.31** Top row: frequency domain filters. Bottom row: corresponding filtered images obtained using Eq. (4.7-1). We used $a = 0.85$ in (c) to obtain (f) (the height of the filter itself is 1). Compare (f) with Fig. 4.29(a).
Effects of Zero-Padding

- Smoothing, no padding
- Smoothing, padded

No padding

Padded
Spatial Zero-Padding for the Filter

FIGURE 4.34
(a) Original filter specified in the (centered) frequency domain.
(b) Spatial representation obtained by computing the IDFT of (a).
(c) Result of padding (b) to twice its length (note the discontinuities).
(d) Corresponding filter in the frequency domain obtained by computing the DFT of (c). Note the ringing caused by the discontinuities in (c). (The curves appear continuous because the points were joined to simplify visual analysis.)
Summary: Filtering in the Frequency Domain

1. For $f(x, y)$, find $P = 2M$, $Q = 2N$

2. Form a padded image $f_p(x, y)$

3. Centering: Multiply $f_p(x, y)$ by $(-1)^{x+y}$

4. Compute DFT $F(u, v)$

5. Generate a real symmetric filter $H(u, v)$ of a size $P \times Q$, centered at $(P/2, Q/2)$ and make $G(u, v) = H(u, v)F(u, v)$

6. Obtain the processed image (should be real in theory)

$$g_p(x, y) = \{\text{real}[g(x, y)]\}(-1)^{x+y}$$

7. Obtained the final processed results $g(x, y)$ from the top left $M \times N$ region
An Example

**FIGURE 4.36**
(a) An $M \times N$ image, $f$.
(b) Padded image, $f_p$ of size $P \times Q$.
(c) Result of multiplying $f_p$ by $(-1)^{x+y}$.
(d) Spectrum of $F_p$.
(e) Centered Gaussian lowpass filter, $H$, of size $P \times Q$.
(f) Spectrum of the product $HF_p$.
(g) $g_p$, the product of $(-1)^{x+y}$ and the real part of the IDFT of $HF_p$.
(h) Final result, $g$, obtained by cropping the first $M$ rows and $N$ columns of $g_p$. 
Correspondence to the Spatial Domain Filter

The FT of a Gaussian function is still a Gaussian function

**FIGURE 4.37**
(a) A 1-D Gaussian lowpass filter in the frequency domain. (b) Spatial lowpass filter corresponding to (a). (c) Gaussian highpass filter in the frequency domain. (d) Spatial highpass filter corresponding to (c). The small 2-D masks shown are spatial filters we used in Chapter 3.
An Example (Sobel Mask)

FIGURE 4.39
(a) A spatial mask and perspective plot of its corresponding frequency domain filter. (b) Filter shown as an image. (c) Result of filtering Fig. 4.38(a) in the frequency domain with the filter in (b). (d) Result of filtering the same image with the spatial filter in (a). The results are identical.
**Image Smoothing Using Frequency Domain Filters – Ideal Lowpass Filter**

**FIGURE 4.40** (a) Perspective plot of an ideal lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.
Locating the Cut-Off Frequency

**FIGURE 4.41** (a) Test pattern of size 688 × 688 pixels, and (b) its Fourier spectrum. The spectrum is double the image size due to padding but is shown in half size so that it fits in the page. The superimposed circles have radii equal to 10, 30, 60, 160, and 460 with respect to the full-size spectrum image. These radii enclose 87.0, 93.1, 95.7, 97.8, and 99.2% of the padded image power, respectively.
Applying the ILPF – Blurring and Ringing

<table>
<thead>
<tr>
<th>Original</th>
<th>ILPF, cutoff 10, Energy 87%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ILPF, cutoff 30 Energy 93.1%</td>
</tr>
<tr>
<td></td>
<td>ILPF, cutoff 60 Energy 95.7%</td>
</tr>
<tr>
<td></td>
<td>ILPF, cutoff 160 Energy 97.8%</td>
</tr>
<tr>
<td></td>
<td>ILPF, cutoff 460 Energy 99.2%</td>
</tr>
</tbody>
</table>
Why?

**FIGURE 4.43**
(a) Representation in the spatial domain of an ILPF of radius 5 and size \(1000 \times 1000\).
(b) Intensity profile of a horizontal line passing through the center of the image.
Butterworth Lowpass Filters (BLPF)

\[ H(u, v) = \frac{1}{1 + \left[ \frac{D(u, v)}{D_0} \right]^{2n} } \]

\( D_0 \) is the cutoff frequency

**FIGURE 4.44** (a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.
Applying the BLPF

Order 2

ILPF, cutoff 10
Energy 87%

ILPF, cutoff 30
Energy 93.1%

ILPF, cutoff 60
Energy 95.7%

ILPF, cutoff 160
Energy 97.8%

ILPF, cutoff 460
Energy 99.2%

Figure 4.45 (a) Original image. (b)–(f) Results of filtering using BLPFs of order 2, with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Fig. 4.42.
**Different-Order BLPF**

**FIGURE 4.46** (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding intensity profiles through the center of the filters (the size in all cases is $1000 \times 1000$ and the cutoff frequency is 5). Observe how ringing increases as a function of filter order.
Gaussian Lowpass Filters

\[ H(u, v) = e^{-D^2(u,v)/2D_0^2} \]

**FIGURE 4.47** (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of \( D_0 \).
Applying the GLPF

D₀ = cutoff radius

FIGURE 4.45 (a) Original image, (b)–(f) Results of filtering using BLPFs of order 2, with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Fig. 4.42.

FIGURE 4.48 (a) Original image, (b)–(f) Results of filtering using GLPFs with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Figs. 4.42 and 4.45.
Summary of Lowpass Filters

**TABLE 4.4**
Lowpass filters. $D_0$ is the cutoff frequency and $n$ is the order of the Butterworth filter.

<table>
<thead>
<tr>
<th>Ideal</th>
<th>Butterworth</th>
<th>Gaussian</th>
</tr>
</thead>
</table>
| $H(u, v) = \begin{cases} 
1 & \text{if } D(u, v) \leq D_0 \\
0 & \text{if } D(u, v) > D_0
\end{cases}$ | $H(u, v) = \frac{1}{1 + [D(u, v)/D_0]^{2n}}$ | $H(u, v) = e^{-D^2(u,v)/2D_0^2}$ |
Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.
FIGURE 4.50  (a) Original image (784 × 732 pixels).  (b) Result of filtering using a GLPF with $D_0 = 100$.  (c) Result of filtering using a GLPF with $D_0 = 80$.  Note the reduction in fine skin lines in the magnified sections in (b) and (c).
Applications

**FIGURE 4.51** (a) Image showing prominent horizontal scan lines. (b) Result of filtering using a GLPF with $D_0 = 50$. (c) Result of using a GLPF with $D_0 = 20$. (Original image courtesy of NOAA.)
Highpass Filters in Frequency Domain

\[ H_{HP}(u, v) = 1 - H_{LP}(u, v) \]
# Highpass Filters

<table>
<thead>
<tr>
<th>Ideal</th>
<th>Butterworth</th>
<th>Gaussian</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H(u, v) = \begin{cases} 0 &amp; \text{if } D(u, v) \leq D_0 \ 1 &amp; \text{if } D(u, v) &gt; D_0 \end{cases}$</td>
<td>$H(u, v) = \frac{1}{1 + [D_0/D(u, v)]^{2n}}$</td>
<td>$H(u, v) = 1 - e^{-D^2(u,v)/2D_0^2}$</td>
</tr>
</tbody>
</table>

TABLE 4.5

Highpass filters. $D_0$ is the cutoff frequency and $n$ is the order of the Butterworth filter.
Highpass Filters In Spatial Domain

**FIGURE 4.53** Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.
Applying IHPF

**FIGURE 4.54** Results of highpass filtering the image in Fig. 4.41(a) using an IHPF with $D_0 = 30, 60, \text{ and } 160$. 
Applying BHPF/GHPF
Applications

**FIGURE 4.57** (a) Thumb print. (b) Result of highpass filtering (a). (c) Result of thresholding (b). (Original image courtesy of the U.S. National Institute of Standards and Technology.)
## Bandreject and Bandpass Filters

**TABLE 4.6**
Bandreject filters. $W$ is the width of the band, $D$ is the distance $D(u, v)$ from the center of the filter, $D_0$ is the cutoff frequency, and $n$ is the order of the Butterworth filter. We show $D$ instead of $D(u, v)$ to simplify the notation in the table.

<table>
<thead>
<tr>
<th>ideal</th>
<th>butterworth</th>
<th>gaussian</th>
</tr>
</thead>
</table>
| $H(u, v) = \begin{cases} 
0 & \text{if } D_0 - \frac{W}{2} \leq D \leq D_0 + \frac{W}{2} \\
1 & \text{otherwise}
\end{cases}$ | $H(u, v) = \frac{1}{1 + \left[ \frac{DW}{D^2 - D_0^2} \right]^{2n}}$ | $H(u, v) = 1 - e^{-\left[ \frac{u^2 - D_0^2}{DW} \right]^2}$ |

![Figure 4.63](a) Bandreject Gaussian filter. (b) Corresponding bandpass filter. The thin black border in (a) was added for clarity; it is not part of the data.
Example

**FIGURE 4.65**

(a) 674 × 674 image of the Saturn rings showing nearly periodic interference.
(b) Spectrum: The bursts of energy in the vertical axis near the origin correspond to the interference pattern.
(c) A vertical notch reject filter.
(d) Result of filtering. The thin black border in (c) was added for clarity; it is not part of the data.

(Original image courtesy of Dr. Robert A. West, NASA/JPL.)
Example

FIGURE 4.66
(a) Result (spectrum) of applying a notch pass filter to the DFT of Fig. 4.65(a).
(b) Spatial pattern obtained by computing the IDFT of (a).
Implementation

Separability of the 2D DFT

Computing the IDFT using a DFT algorithm

The Fast Fourier Transform (read Chapter 4.11.3)

FIGURE 4.67
Computational advantage of the FFT over a direct implementation of the 1-D DFT. Note that the advantage increases rapidly as a function of $n$. 
Reading Assignments

Chapter 4.3 – 4.11