Today's Agenda

Affine transformation

Changing Representations

Any point or vector has a representation in a frame

$\mathbf{a}=[\alpha_1\ \alpha_2\ \ \alpha_3\ \alpha_4\ ]$ in the first frame
$\mathbf{b}=[\beta_1\ \beta_2\ \ \beta_3\ \beta_4\ ]$ in the second frame

where $\alpha_4 = \beta_4 = 1$ for points and $\alpha_4 = \beta_4 = 0$ for vectors

We can change the representation from one frame to the other as

$$\mathbf{a}=\mathbf{M}^T\mathbf{b} \quad \text{and} \quad \mathbf{b}=(\mathbf{M}^T)^{-1}\mathbf{a}$$

The matrix $\mathbf{M}$ is 4 x 4 and specifies an affine transformation in homogeneous coordinates

Affine Transformations

Every linear transformation is equivalent to a change in frames

Every affine transformation preserves lines: a line in a frame transforms to a line in another frame

An affine transformation

- Characteristic of many physically important transformations
  - Rigid body transformations: rotation, translation
  - Scaling, shear

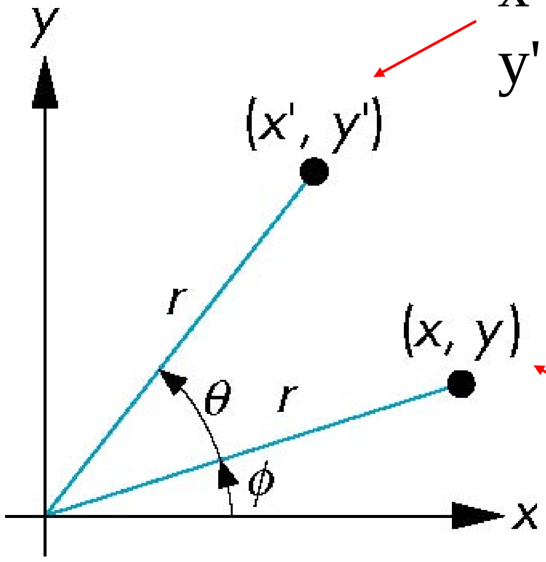
- has only 12 *degrees of freedom* because 4 of the elements in the matrix are fixed

- are a subset of all possible 4 x 4 linear transformations

Rotation in 2D

Consider rotation about the origin by θ degrees

- radius stays the same, angle increases by θ



$$x' = r \cos (\phi + \theta)$$
$$y' = r \sin (\phi + \theta)$$

$$x' = x \cos \theta - y \sin \theta$$
$$y' = x \sin \theta + y \cos \theta$$

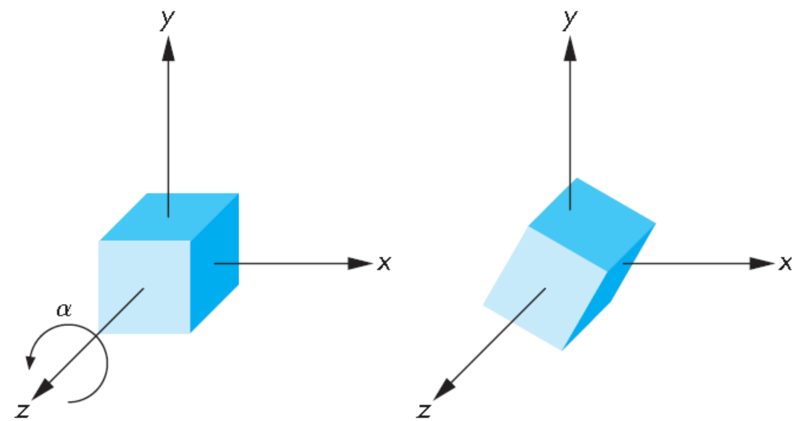
$$x = r \cos \phi$$
$$y = r \sin \phi$$

Rotation about the z axis

Rotation about z axis in three dimensions leaves all points with the same z

- Equivalent to rotation in two dimensions in planes of constant z

$$x' = x\cos\alpha - y\sin\alpha$$
$$y' = x\sin\alpha + y\cos\alpha$$
$$z' = z$$



- or in homogeneous coordinates

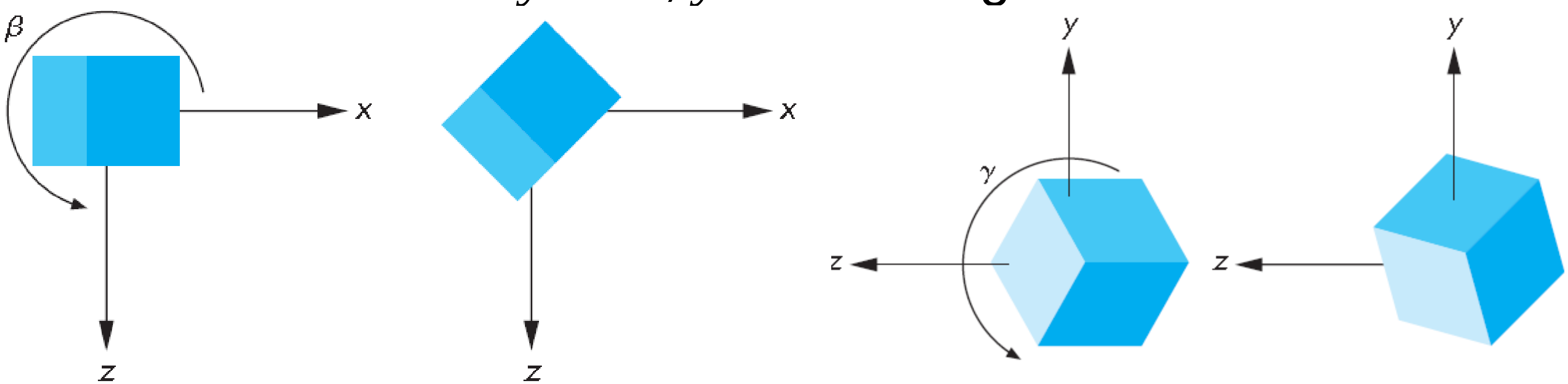
$$\mathbf{p}' = \mathbf{R}_z(\alpha)\mathbf{p}$$

Rotation Matrix ← $\mathbf{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Rotation about x and y axes

Same argument as for rotation about *z* axis

- For rotation about x axis, x is unchanged
- For rotation about y axis, y is unchanged



$$\mathbf{R}_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma & 0 \\ 0 & \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverses

Although we could compute inverse matrices by general formulas, we can use simple geometric observations

- Translation: $\mathbf{T}^{-1}(d_x, d_y, d_z) = \mathbf{T}(-d_x, -d_y, -d_z)$
- Rotation: $\mathbf{R}^{-1}(\theta) = \mathbf{R}(-\theta)$
  - Holds for any rotation matrix
  - Note that since $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$

  $\mathbf{R}^{-1}(\theta) = \mathbf{R}^{T}(\theta) \longrightarrow \mathbf{R}\mathbf{R}^{T} = \mathbf{R}\mathbf{R}^{-1} = I$ Rotation matrix is orthonormal matrix
- Scaling: $\mathbf{S}^{-1}(s_x, s_y, s_z) = \mathbf{S}(1/s_x, 1/s_y, 1/s_z)$

Multiple Transformations

We can form arbitrary affine transformation matrices by multiplying rotation, translation, and scaling matrices

Intuitive way: $\mathbf{p'}=\mathbf{M}_3[\mathbf{M}_2(\mathbf{M}_1\mathbf{p})]$ Pre-multiply

Alternative way: $\mathbf{p'}=(\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1)\mathbf{p}$ Post-multiply

Which one is better?

The same transformation is applied to many vertices,

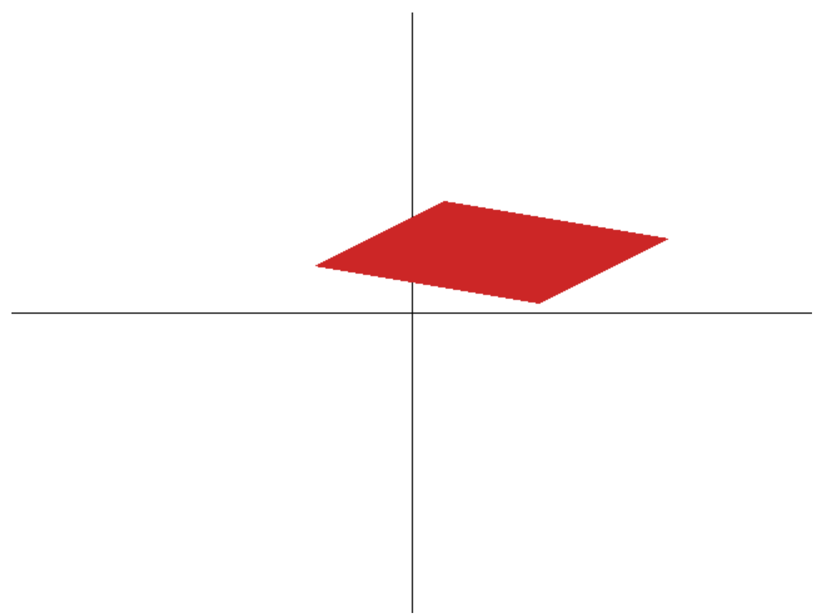
- the matrix $\mathbf{M}=\mathrm{M}_3\mathrm{M}_2\mathrm{M}_1$ can be precomputed
- the computational cost of **M** can be ignored compared to the cost of computing $\mathbf{Mp}$ for many vertices $\mathbf{p}$

Exercise: Composing Transformations

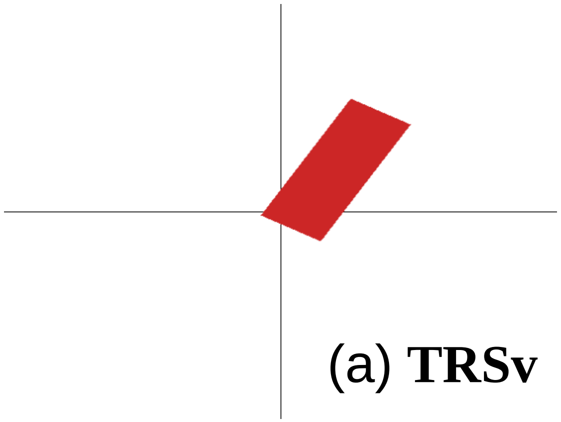
original | **R** = rotate(60°) | **S** = scale(1.3, 0.5) | **T** = trans(0.2, 0.2)

What order of **R**, **S**, **T** will produce this figure?

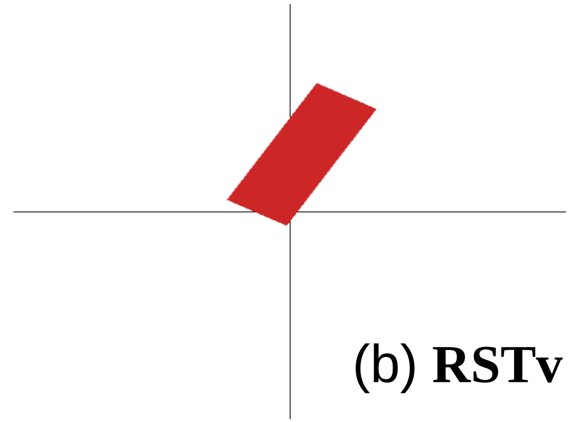
(a) **TRSv**
(b) **RSTv**
(c) **TSRv**
(d) **RTSv**



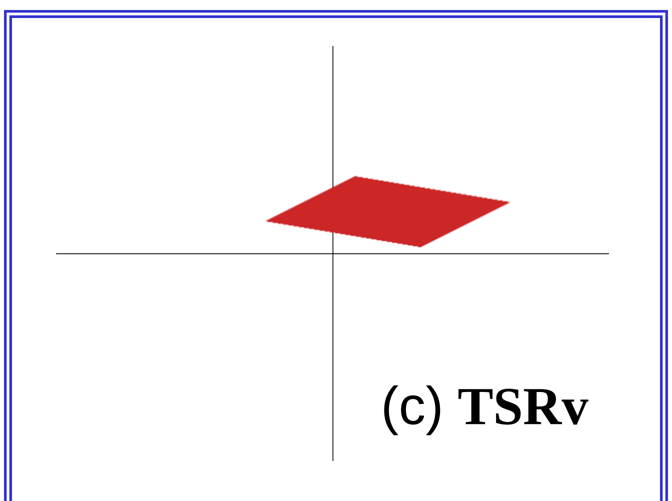
Exercise: Composing Transformations



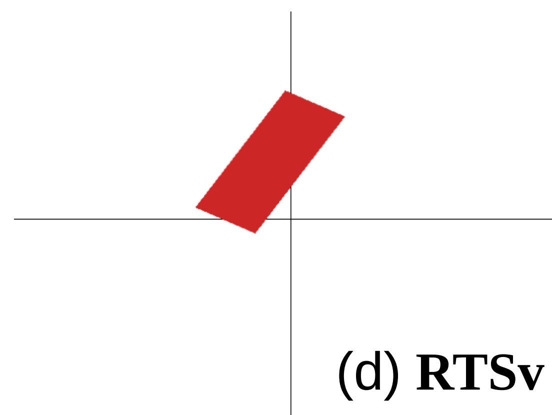
(a) **TRSv**



(b) **RSTv**



(c) **TSRv**



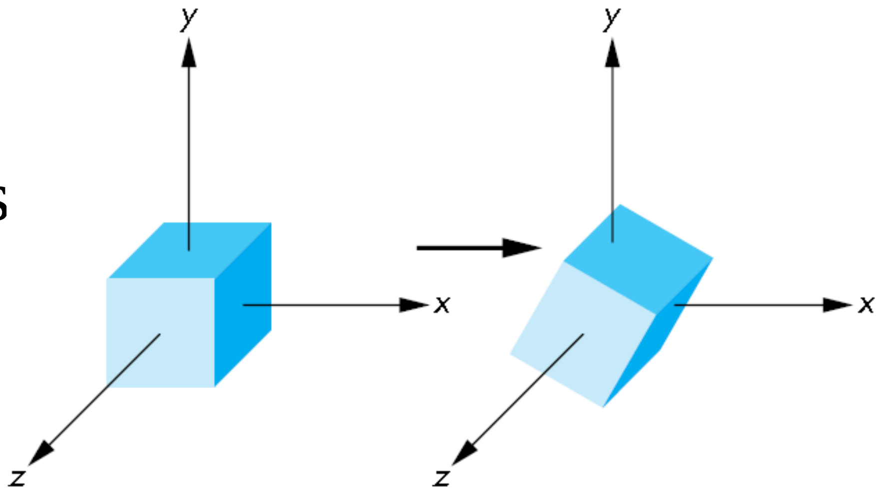
(d) **RTSv**

General Rotation About the Origin

A general rotation about the origin can be decomposed into successive of rotations about the *x*, *y*, and *z* axes

$$\mathbf{R} = \mathbf{R}_z(\alpha)\, \mathbf{R}_y(\beta)\, \mathbf{R}_x(\gamma)$$

α, β, γ are called the Euler angles



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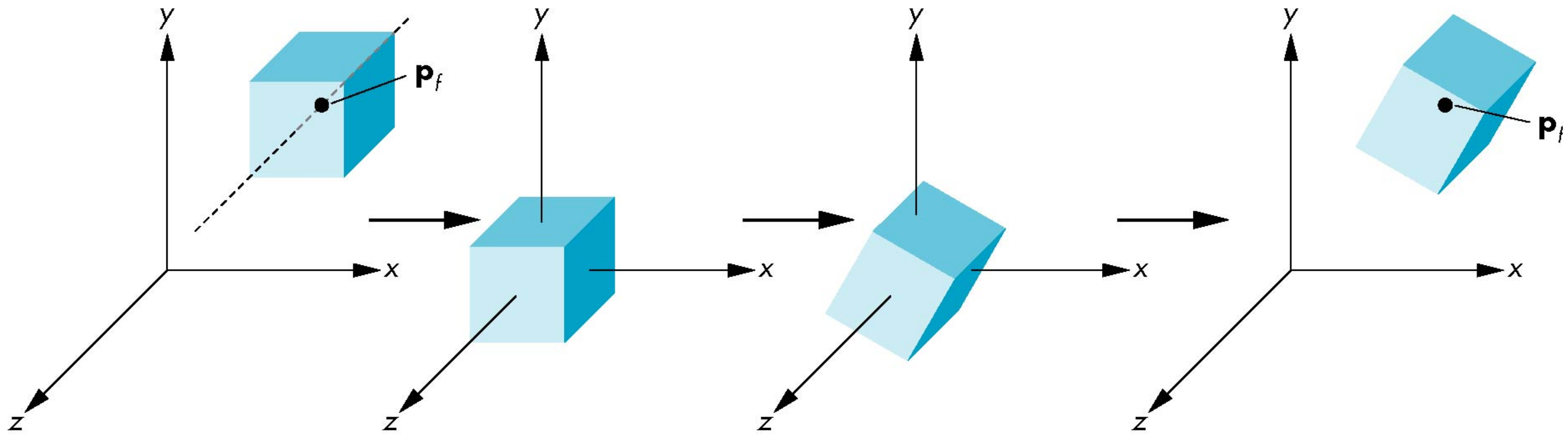
Important:

- **R** is unique
- For a given order, rotations do not commute
- We can use rotations in another order but with different angles

Rotation About a Fixed Point Other than the Origin

- Move fixed point to origin
- Rotate around the origin
- Move fixed point back



Instancing

How do we describe multiple object in a scene?

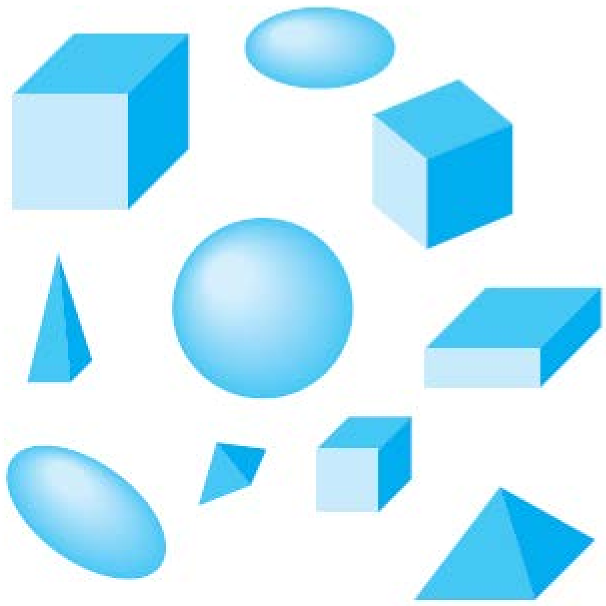
Intuitive solution:

Specify the vertices for each object

A better solution:

Specify a set of simple objects with

- a convenient size,
- a convenient location,
- a convenient orientation



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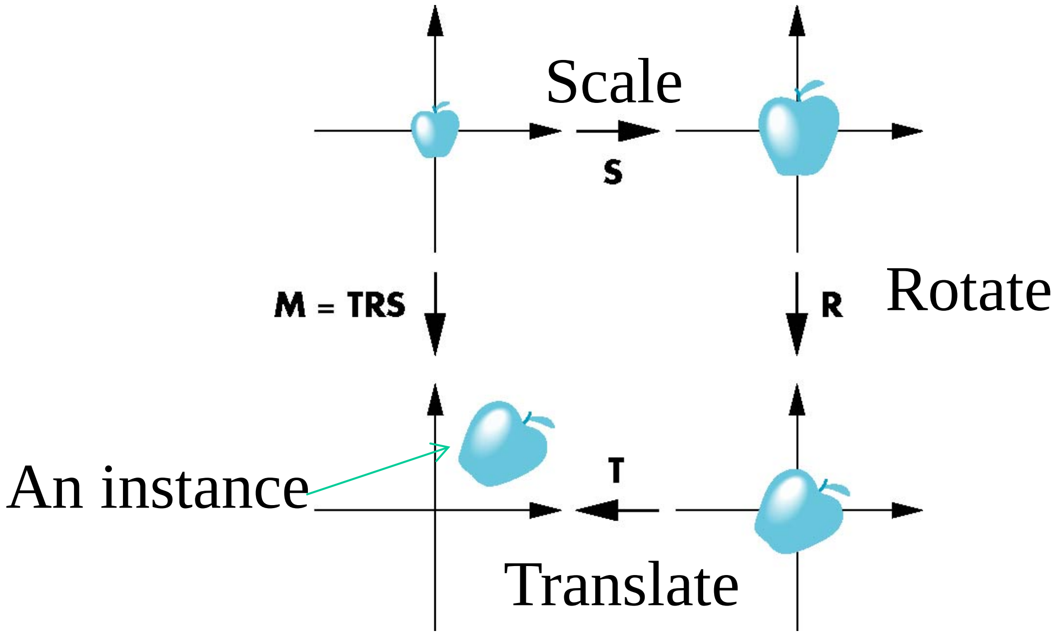
Instancing

In modeling, we often start with a simple object centered at the origin, oriented with the axis, and at a standard size

An occurrence of this object is an **instance** of the object class

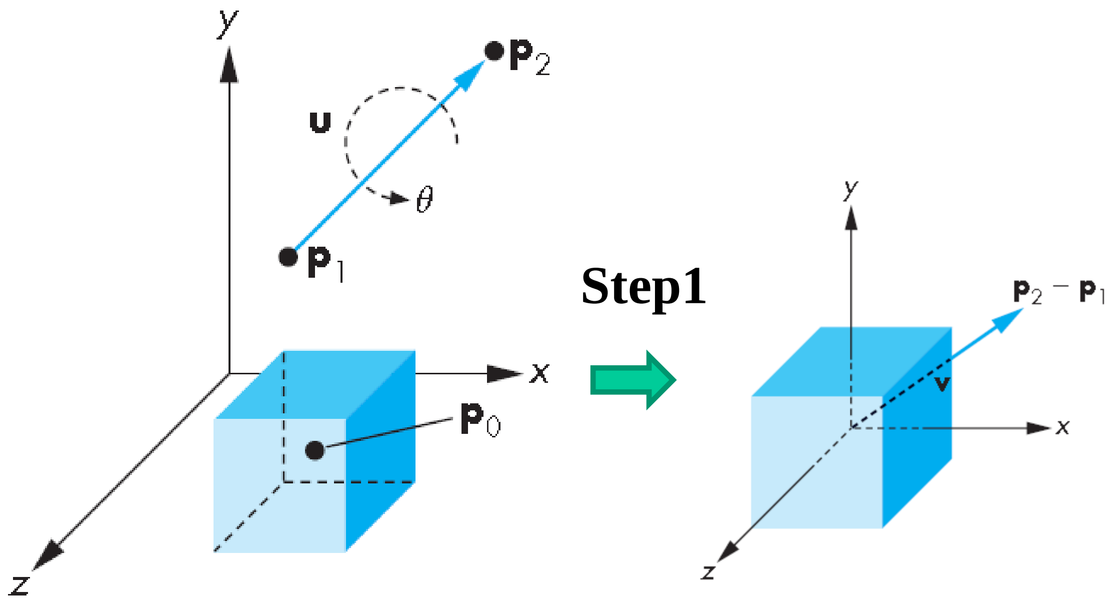
We apply an ***instance transformation*** to its vertices to



General Rotation about An Arbitrary Vector

How do we achieve a rotation θ about an arbitrary vector?

Step 1: move the fixed point to the origin $\mathbf{M}_1 = \mathbf{T}(-\mathbf{p}_0)$



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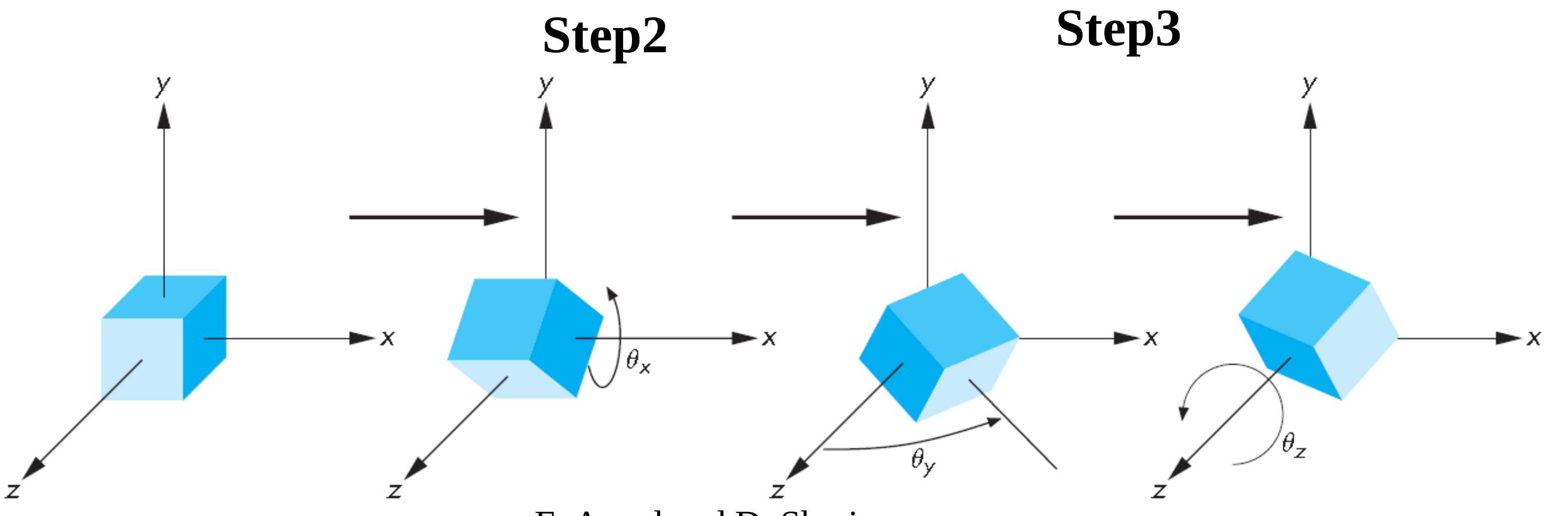
General Rotation about An Arbitrary Vector

Step 2: align the arbitrary vector $\mathbf{v}=\frac{\mathbf{p}_2-\mathbf{p}_1}{|\mathbf{p}_2-\mathbf{p}_1|}$ with the z-axis by two rotations about the x-axis and y-axis with θ_x and θ_y, respectively

$$\mathbf{M}_2 = \mathbf{R}_y(\theta_y)\mathbf{R}_x(\theta_x)$$

Step 3: rotate by θ about the z-axis

$$\mathbf{M}_3 = \mathbf{R}_z(\theta)$$



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General Rotation about An Arbitrary Vector

Step 4: undo the two rotations for aligning z-axis

$$\mathbf{M}_4 = \mathbf{R}_x(-\theta_\mathrm{x})\mathbf{R}_y(-\theta_\mathrm{y})$$

Step 5: move the fixed point back

$$\mathbf{M}_5 = \mathbf{T}(\mathbf{p}_0)$$

The overall transformation matrix is

$$\mathbf{M} = \mathbf{M}_5\mathbf{M}_4\mathbf{M}_3\ \mathbf{M}_2\ \mathbf{M}_1$$

How to Determine $\boldsymbol{\theta_x}$ and $\boldsymbol{\theta_y}$

Let $\mathbf{v} = [\alpha_x \quad \alpha_y \quad \alpha_z]^T$ **and** $\alpha_x{}^2 + \alpha_y{}^2 + \alpha_z{}^2 = 1$

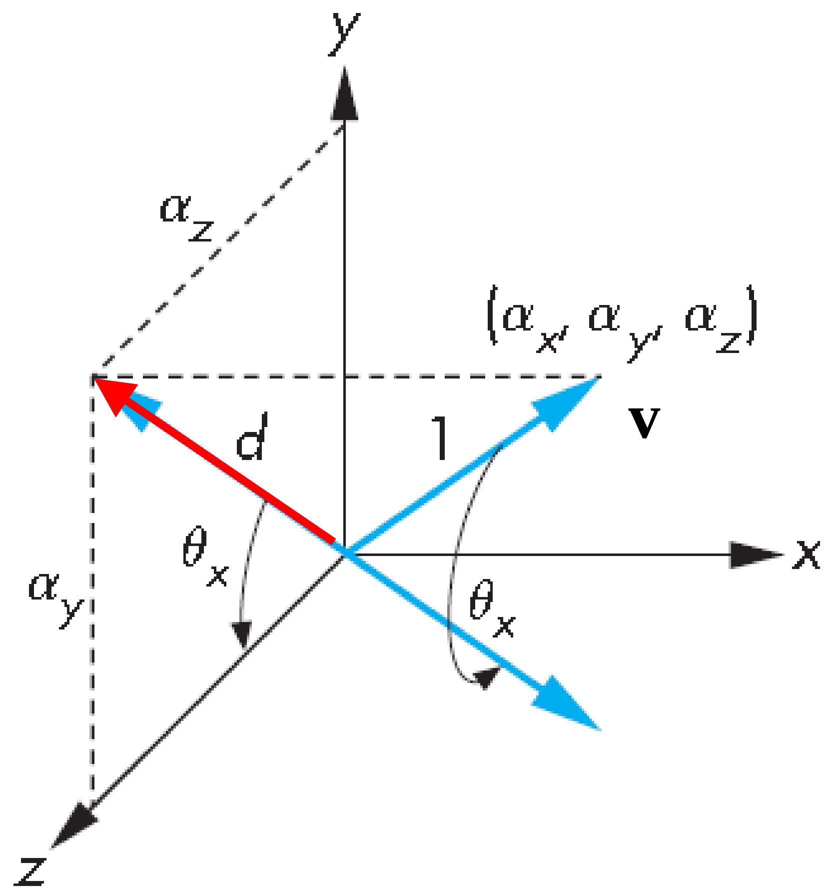
Compute $\mathbf{R}_x(\boldsymbol{\theta}_\mathrm{x})$

$$\cos\theta_\mathrm{x} = \frac{\alpha_z}{d} \quad \text{and} \quad \sin\theta_\mathrm{x} = \frac{\alpha_y}{d}$$

$$\mathbf{R}_x(\theta_\mathrm{x}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\alpha_z}{d} & -\frac{\alpha_y}{d} & 0 \\ 0 & \frac{\alpha_y}{d} & \frac{\alpha_z}{d} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

d is the projection of **v** on the y-z plane

$$d = \sqrt{\alpha_y{}^2 + \alpha_z{}^2}$$



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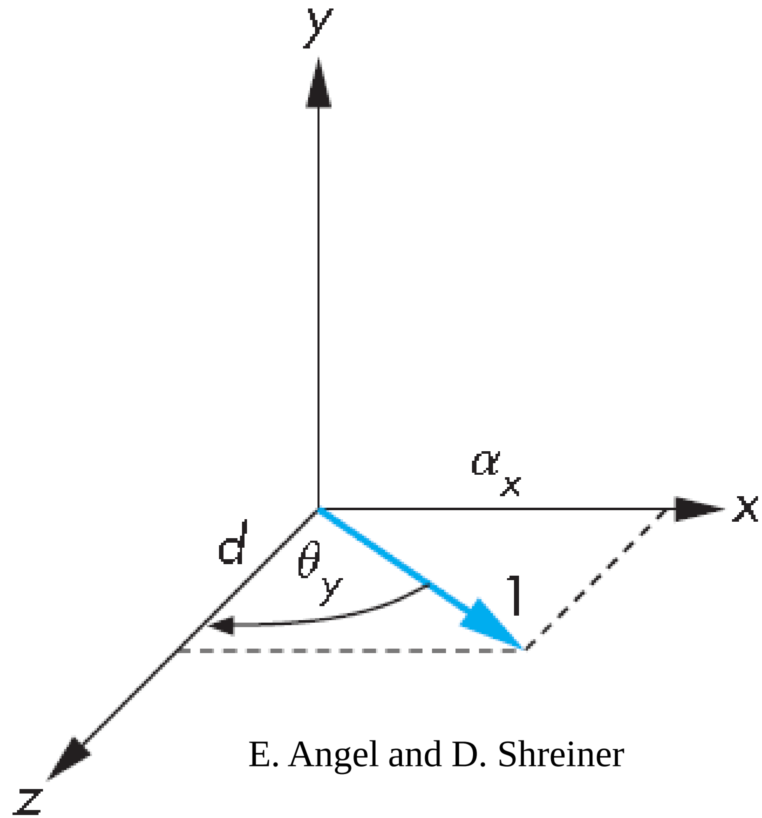
How to Determine $\boldsymbol{\theta_x}$ and $\boldsymbol{\theta_y}$

Compute $\mathbf{R}_y(\boldsymbol{\theta}_\mathrm{y})$

$$\cos\theta_\mathrm{y} = d \quad \text{and} \quad \sin\theta_\mathrm{y} = -\alpha_x$$

$$\mathbf{R}_y(\theta_\mathrm{y}) = \begin{bmatrix} d & 0 & -\alpha_x & 0 \\ 0 & 1 & 0 & 0 \\ \alpha_x & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



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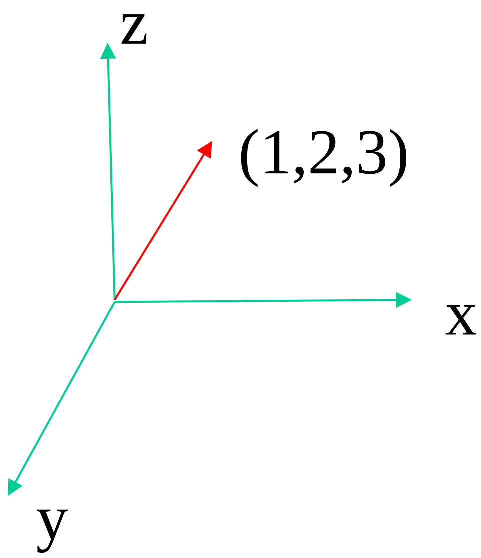
An Example

Problem: rotate an object by 45 degrees about the line passing through the origin and the point (1,2,3)

Step1: Normalize the vector for rotation

$$\mathbf{p} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{v} = \frac{\mathbf{p}}{|\mathbf{p}|} = \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \\ 0 \end{bmatrix}$$

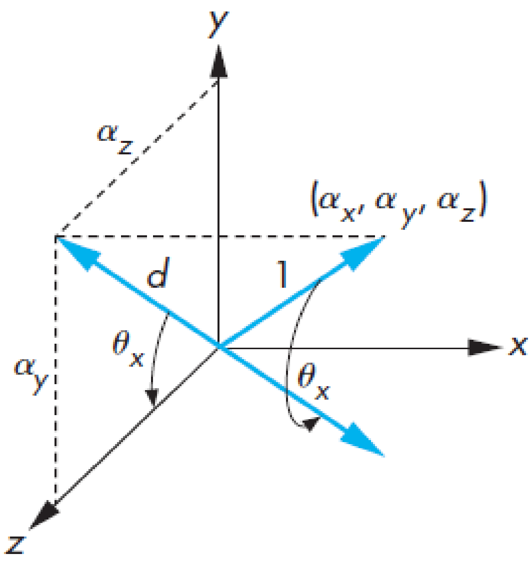


An Example (Cont'd)

$$\alpha_x = \frac{1}{\sqrt{14}}, \alpha_y = \frac{2}{\sqrt{14}}, \alpha_z = \frac{3}{\sqrt{14}},$$

Step2: rotate about the x-axis about θ_x

Calculate the angle θ_x



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$$\cos\theta_x = \frac{\alpha_z}{d} = \frac{\alpha_z}{\sqrt{{\alpha_y}^2 + {\alpha_z}^2}}$$

$$\cos\theta_x = \frac{\alpha_z}{d} = \frac{3}{\sqrt{13}}$$

An Example (Cont'd)

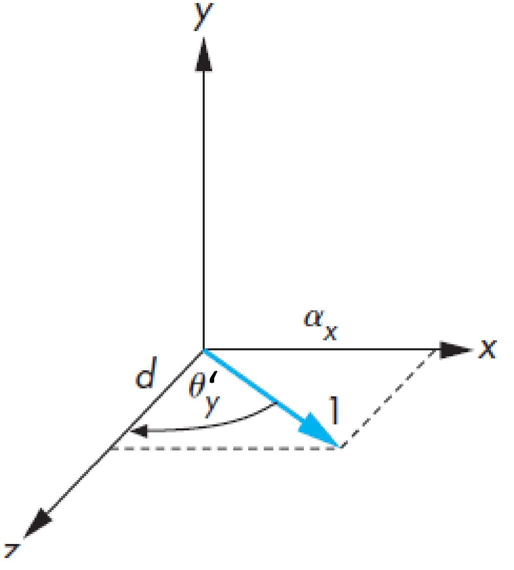
$$\alpha_x = \frac{1}{\sqrt{14}}, \alpha_y = \frac{2}{\sqrt{14}}, \alpha_z = \frac{3}{\sqrt{14}},$$

Step3: rotate about the y-axis about $\theta_y = -\theta_y{}'$

Calculate the angle $\theta_y{}'$

$$\cos\theta_y{}' = d = \sqrt{\alpha_y{}^2 + \alpha_z{}^2}$$

$$\cos\theta_y{}' = \sqrt{\frac{13}{14}}$$



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An Example (Cont'd)

Step4: rotate about the z-axis about 45 degrees

Step5: rotate about the y-axis about $-\theta_y$

Step6: rotate about the x-axis about $-\theta_x$

$$R = R_x\left(-\cos^{-1}\frac{3}{\sqrt{13}}\right) R_y\left(\cos^{-1}\sqrt{\frac{13}{14}}\right) R_z(45) R_y\left(-\cos^{-1}\sqrt{\frac{13}{14}}\right)$$

$$R_x\left(\cos^{-1}\frac{3}{\sqrt{13}}\right)$$

$$= \begin{bmatrix} \frac{2+13\sqrt{2}}{28} & \frac{2-\sqrt{2}-3\sqrt{7}}{14} & \frac{6-3\sqrt{2}+4\sqrt{7}}{28} & 0 \\ \frac{2-\sqrt{2}+3\sqrt{7}}{14} & \frac{4+5\sqrt{2}}{14} & \frac{6-3\sqrt{2}-\sqrt{7}}{14} & 0 \\ \frac{6-3\sqrt{2}-4\sqrt{7}}{28} & \frac{6-3\sqrt{2}+\sqrt{7}}{14} & \frac{18+5\sqrt{2}}{28} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$