Smooth Functions

- **Eventually nondecreasing function:**

$$f(n_1) < f(n_2), \qquad \text{for} \quad n_0 \le n_1 < n_2$$

e.g., *n, logn,* n^2, 2^n Is sin(n) eventually nondecreasing?

- **Smooth function:**

f(n) is eventually nondecreasing and $f(2n) \in \Theta(f(n))$

- *f(n)* cannot grow too fast, e.g.*, n, logn,* n^2
- 2^n, n! are not smooth functions

Properties of Smooth Functions

- If $f(n)$ is a smooth function, for any constant integer $b \geq 2$

  $f(bn) \in \Theta(f(n))$ **(See Appendix B for the proof)**

- **Smoothness rule:**

  $T(n)$ is eventually non decreasing and $f(n)$ is a smooth function

  If $T(n) \in \Theta(f(n))$ for $n = b^k, b \geq 2$

  $$\text{then} \quad T(n) \in \Theta(f(n)) \text{ for every } n$$

  - Analogous results hold for big O and big Ω

  (See Appendix B for the proof)

A General Divide-and-Conquer Recurrence: Master Theorem

$T(n)$ **is an eventually nondecreasing function**

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$ **where** $f(n) \in \Theta(n^d), a \geq 1, b \geq 2, c > 0, d \geq 0$

$T(1) = c$ **-- General Divide-and-Conquer Recurrence**

Closed form solution: $$T(n) = n^{\log_b a}\left[T(1) + \sum_{j=1}^{\log_b n} \frac{f(b^j)}{a^j}\right]$$

$a < b^d$	$T(n) \in \Theta(n^d)$
$a = b^d$	$T(n) \in \Theta(n^d \log n)$
$a > b^d$	$T(n) \in \Theta(n^{\log_b a})$

Example of Using Master Theorem

$$T(n) = n^{\log_b a}\left[T(1) + \sum_{j=1}^{\log_b n} \frac{f\left(b^j\right)}{a^j}\right]$$

$T(n) = T(n/2) + 1 \Rightarrow a = 1, b = 2, f(n) = 1$

T(1)=2

$$T(n) = n^{\log_2 1}\left[T(1) + \sum_{j=1}^{\log_2 n} \frac{1}{1}\right] = n^0[T(1) + \log_2 n] = 2 + \log_2 n$$

$a = 1, b = 2, d = 0, \Rightarrow a = b^d \Rightarrow T(n) \in \Theta(\log n)$

Example of Using Master Theorem

$$T(n) = n^{\log_b a}\left[T(1) + \sum_{j=1}^{\log_b n} \frac{f(b^j)}{a^j}\right]$$

$T(n) = 2T(n/2) + 3n$ ⟹ $a = 2, b = 2, f(n) = 3n$

T(1)=2

$$T(n) = n^{\log_2 2}\left[T(1) + \sum_{j=1}^{\log_2 n} \frac{3 * 2^j}{2^j}\right] = n^1[T(1) + 3\log_2 n]$$

$$= 2n + 3n\log_2 n$$

$a = 2, b = 2, d = 1,$ ⟹ $a = b^d$ ⟹ $T(n) \in \Theta(n \log n)$

Example of Using Master Theorem

$$T(n) = n^{\log_b a}\left[T(1) + \sum_{j=1}^{\log_b n} \frac{f\left(b^j\right)}{a^j}\right]$$

$T(n) = 3T(n/2) + n$ ⇨ $a = 3, b = 2, f(n) = n$

T(1)=2

⇨ $T(n) = n^{\log_2 3}\left[T(1) + \sum_{j=1}^{\log_2 n} \frac{2^j}{3^j}\right] = n^{\log_2 3}\left[T(1) + \sum_{j=1}^{\log_2 n} \left(\frac{2}{3}\right)^j\right]$

How to calculate $\sum_{j=1}^{\log_2 n} \left(\frac{2}{3}\right)^j$?

In Appendix A, $\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}$ $(a \neq 1)$

Example of Using **Master Theorem**

$$T(n) = n^{\log_b a}\left[T(1) + \sum_{j=1}^{\log_b n} \frac{f(b^j)}{a^j}\right]$$

$T(n) = 3T(n/2) + n$ ⇨ $a = 3, b = 2, f(n) = n$

T(1)=2

⇨ $T(n) = n^{\log_2 3}\left[T(1) + \sum_{j=1}^{\log_2 n} \left(\frac{2}{3}\right)^j\right] \approx 4n^{\log_2 3}$

Order of growth? $\Theta(n^{\log_2 3})$

$a = 3, b = 2, d = 1,$ ⇨ $a > b^d$ ⇨ $T(n) \in \Theta(n^{\log_2 3})$

More Example of Using Master Theorem

$a < b^d$	$T(n) \in \Theta(n^d)$
$a = b^d$	$T(n) \in \Theta(n^d \log n)$
$a > b^d$	$T(n) \in \Theta(\boldsymbol{n}^{\log_b \boldsymbol{a}})$

$T(n) = 2T(n/2) + 1$ ⇨ $a=2$, $b=2$, $d=0$, $a>b^d$ $T(n) \in \Theta(n)$

$T(n) = T(n/2)+n$ ⇨ $a=1$, $b=2$, $d=1$, $a<b^d$ $T(n) \in \Theta(n)$

$T(n) = 3T(n/2)+ n^2$ ⇨ $a=3$, $b=2$, $d=2$, $a<b^d$ $T(n) \in \Theta(n^2)$

Summary: Methods for Solving Recurrence Relations

- Forward substitutions
- Backward substitutions
- Linear 2^{nd} order recurrences with constant coefficients
- Following the solution to important recurrence type if appliable
- Master Theorem for general divide-and-conquer recurrence

Reading Assignment

Chapter 3.3 Closest-pair and Convex-Hull Problems by Brute Force

Chapter 3.4 Exhaustive Search