Breadth-First Search (BFS)

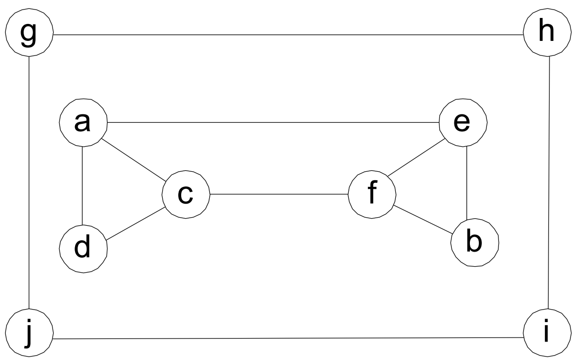
Explore graph moving across to all the neighbors of last visited vertex

Similar to level-by-level tree traversals

Instead of a stack (LIFO), breadth-first uses queue (FIFO)

Applications: same as DFS

BFS Example – undirected graph



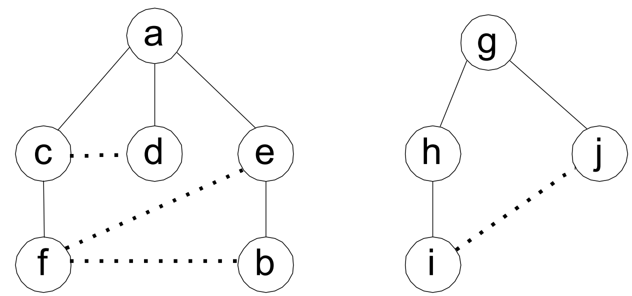
Input Graph

(Adjacency matrix / linked list

a_1 c_2 d_3 e_4 f_5 b_6

g_7 h_8 j_9 i_{10}

Queue



BFS forest

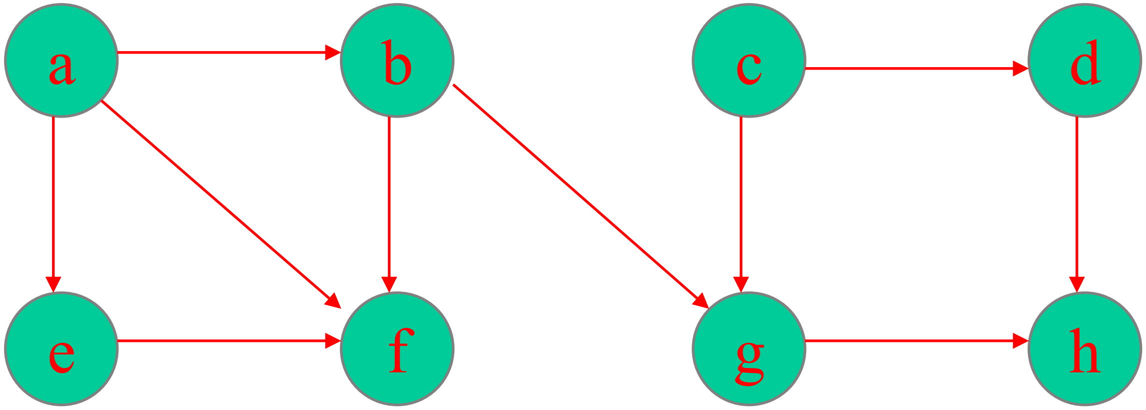
(Tree edge / Cross edge)

BFS algorithm

ALGORITHM BFS(G)
//Input: Graph $G = < V, E >$
//Output: Graph G with its
//vertices marked with
//consecutive integers in the
//order they've been visited by
//BFS traversal
count ← 0
mark each vertex with 0
for each vertex v in V do
    if v is marked with 0
        bfs(v)

bfs(*v*)
count ← *count* + 1
mark *v* with *count*
initialize *queue* with *v*
while *queue* is not empty **do**
    for each vertex *w* adjacent to *the front vertex* **do**
        if *w* is marked with 0
            count ← *count* + 1
            mark *w* with *count*
            add *w* to the end of the *queue*
    remove the front vertex from the *queue*

Example – Directed Graph



BFS traversal:

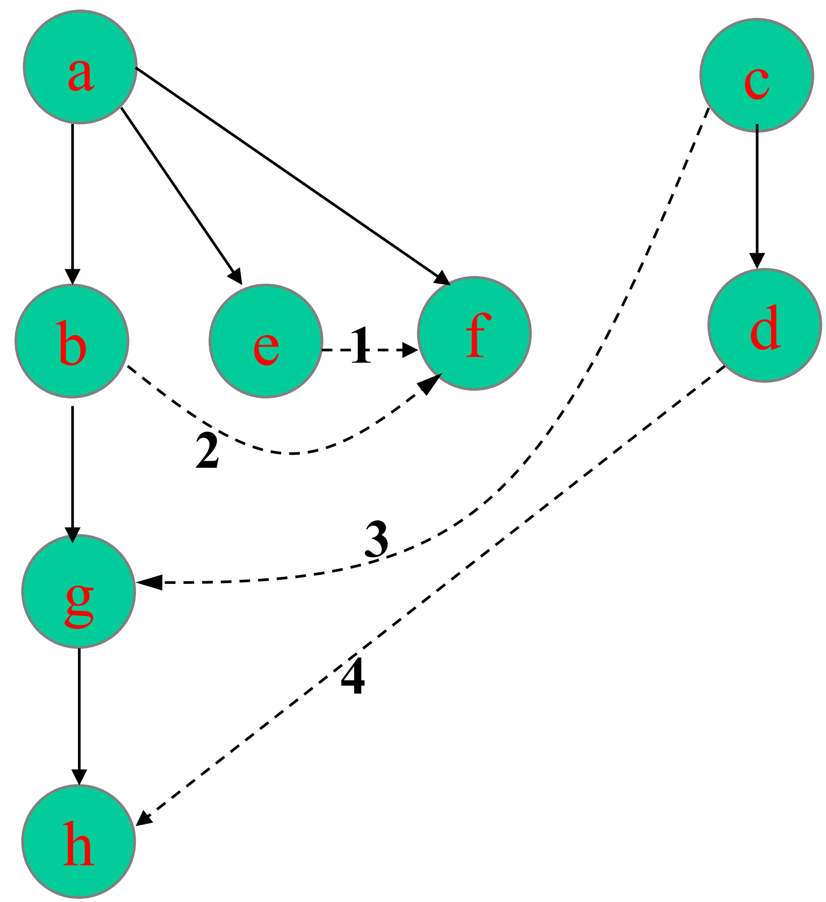
BFS Forest and Queue

a_1
b_2
e_3
f_4
g_5
h_6

c_7
d_8

How many cross edges? 4

Queue



BFS forest

Breadth-first search: Notes

BFS has same efficiency as DFS and can be implemented with graphs represented as:

- Adjacency matrices: $\Theta(|V|^2)$
- Adjacency linked lists: $\Theta(|V|+|E|)$

Yields single ordering of vertices (order added/deleted from queue is the same)

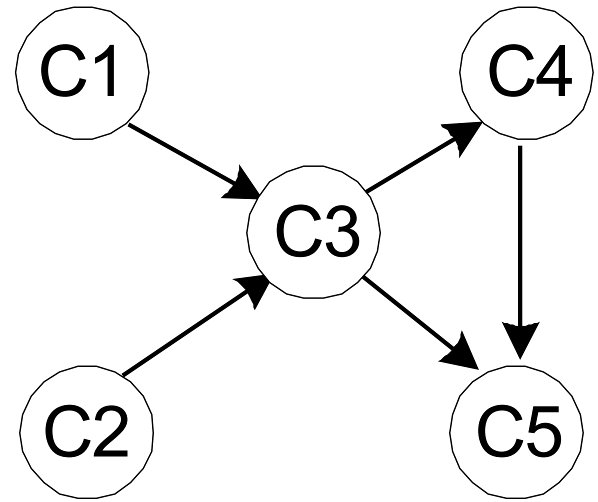
Graph Traversal

- **DFS**
  - Uses a stack
  - Yields two distinct ordering of vertices:
    - Preorder traversal: as vertices are first encountered (pushed onto stack)
    - Postorder traversal: as vertices become dead-ends (popped off stack)
  - Result in a DFS forest
    -- Tree edges, back edges, forward edges, and cross edges

- **BFS**
  - Uses a queue
  - Yields one ordering of vertices
  - Result in a BFS forest with tree edges and cross edges
- **Both DFS and BFS have efficiency**
  - Adjacency matrices: $\Theta(|V|^2)$
  - Adjacency linked lists: $\Theta(|V|+|E|)$

Directed Acyclic Graph (DAG)

A directed graph with no cycles

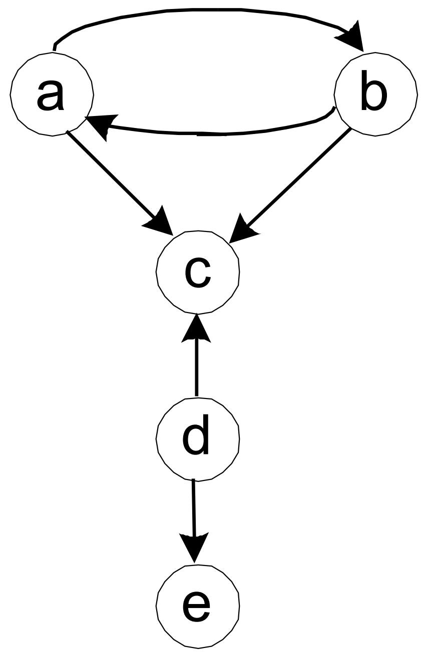


Arise in modeling many problems, eg:

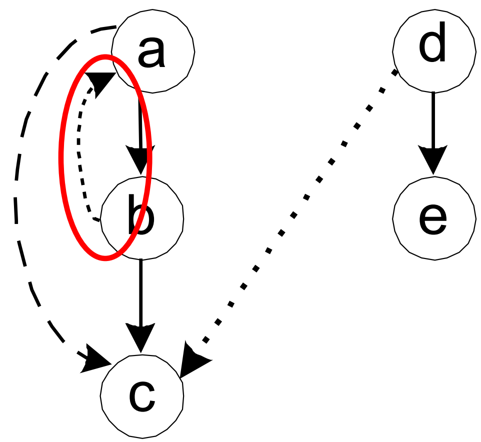
- prerequisite structure
- food chains

A digraph is a DAG if its DFS forest has no back edge.

Example:



(a)
DG



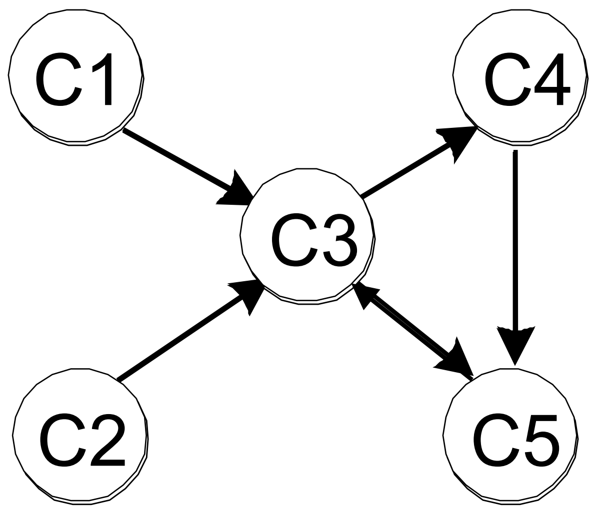
Not a DAG!

(b)
DFS forest

Topological Sorting

Problem: find an order of vertices such that for every edge in the graph, the starting vertex is listed before the ending vertex

Example:



Five courses has the prerequisite relation shown in the left. Find the right order to take all of them sequentially

Note: problem is solvable iff graph is DAG

Topological Sorting Algorithms

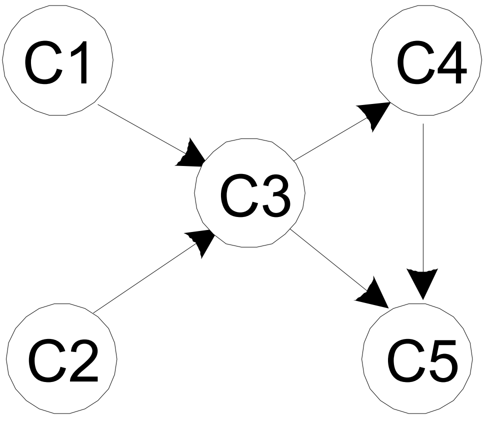
DFS-based algorithm:

- DFS traversal: note the order with which the vertices are popped off stack (dead end)
- Reverse order solves topological sorting
- Back edges encountered?→ NOT a DAG!

Source removal algorithm

- Repeatedly identify and remove a *source* vertex, i.e., a vertex that has no incoming edges

An Example: DFS-based Topological Sorting



(a)

C5 $_1$
C4 $_2$
C3 $_3$
C1 $_4$ C2 $_5$

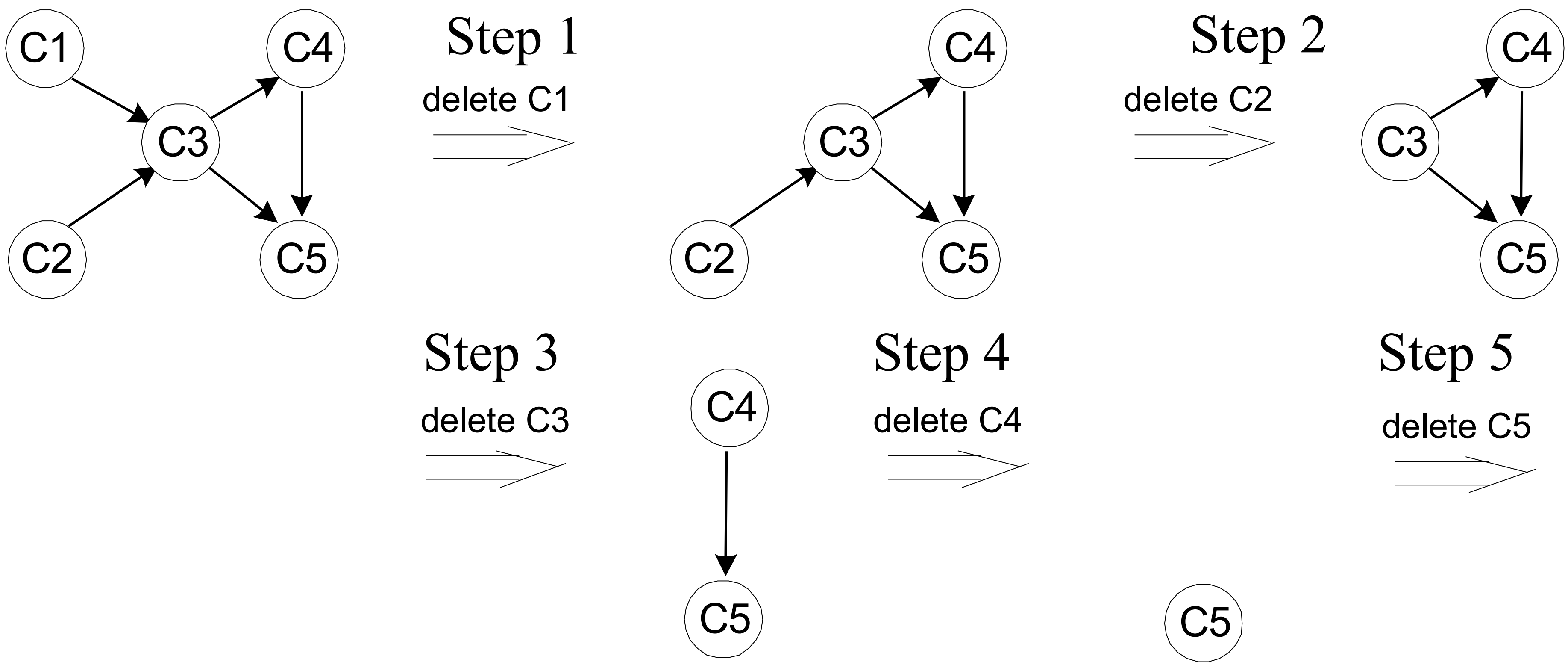
(b)

The popping-off order:
C5, C4, C3, C1, C2
The topologically sorted list:
C2 C1 C3 C4 C5

(c)

Θ(V+E) using adjacency linked lists

An Example: Source removal



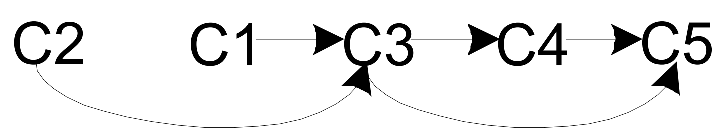
C1 C2 C3 C4 C5

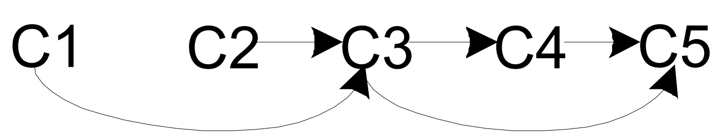
Θ(V+E) using adjacency linked lists

How to implement it?

Comparison

DFS based algorithm and the source removal algorithm may produce different valid topological order lists.

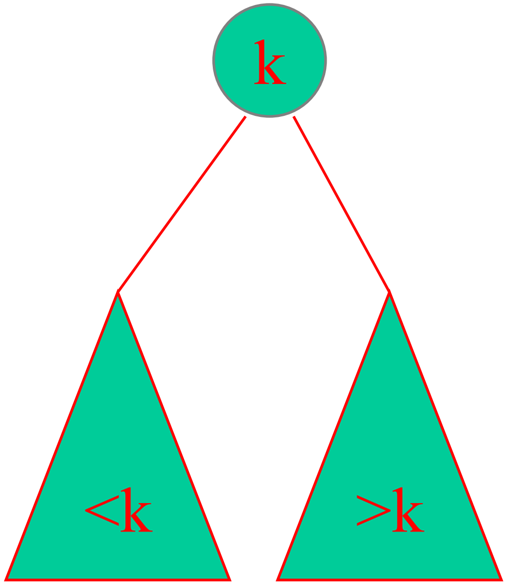




Variable-Size-Decrease: Binary Search Trees

- Every element in the left subtree is smaller than the root
- Every element in the right subtree is larger than the root
- Search a key in a binary search tree is reduced to search in a subtree in each iteration.
- The height of the subtree changes each time

→ variable-size-decrease



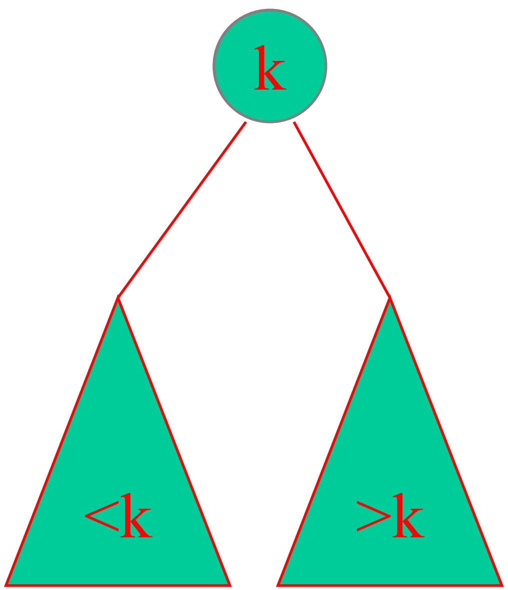
Search a Key in a Binary Search Tree

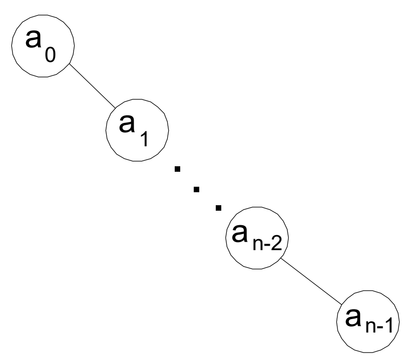
Basic operation: key comparison

# of comparisons in the worst case: h+1

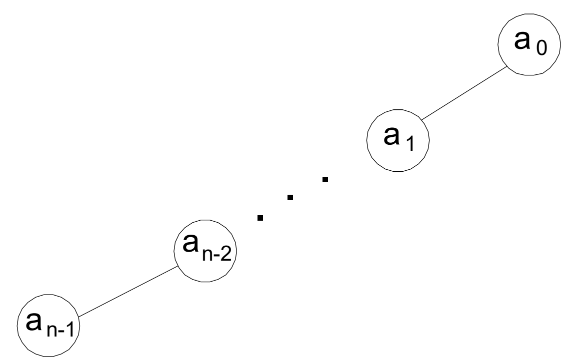
$$\log|V| \leq h \leq |V| - 1$$

Worst case: the tree degrades to a singly linked list Θ(|V|)
Average case: Θ(log|V|)





(a)



(b)

Searching and insertion in binary search trees

Searching – straightforward

Insertion – search for key, insert at leaf where search terminated

Example 1: 5, 10, 3, 1, 7, 12, 9

Example 2: 4, 5, 7, 2, 1, 3, 6

Reading Assignments

Chapter 5.3, 5.4 and 5.5

Now, Chapter 5 -- Divide and Conquer

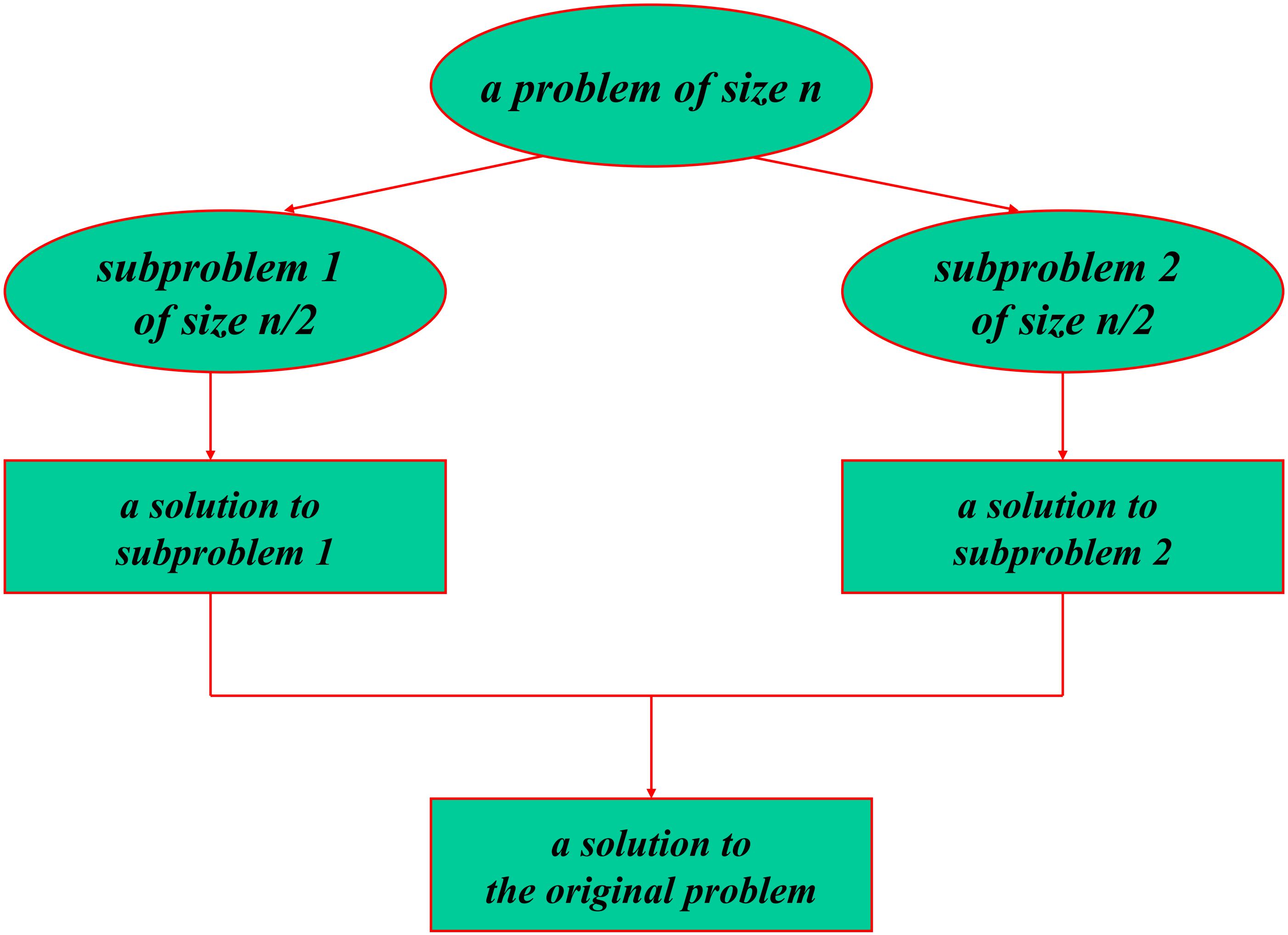
The most well-known algorithm design strategy:

Divide instance of problem into two or more smaller instances of the same problem, ideally of about the same size

Solve smaller instances recursively

Obtain solution to original (larger) instance by combining these solutions obtained for the smaller instances

Divide-and-conquer technique



An Example

Compute the sum of n numbers a_0, a_1, …, a_{n-1}.

Question: How to design a divide-and-conquer algorithm to solve this problem and what is its complexity?

Use divide-and-conquer strategy:

$$a_0 + ... + a_{n-1} = (a_0 + ... + a_{\lfloor n/2 \rfloor - 1}) + (a_{\lfloor n/2 \rfloor} + a_{n-1})$$

What is the recurrence and the complexity of this recursive algorithm?

Does it improve the efficiency of the brute-force algorithm?

General Divide and Conquer Recurrence

$$C(n) = 2C\left(\frac{n}{2}\right) + 1, for\ n > 1 \qquad C(1) = 0$$

$T(n) = aT(n/b) + f(n)$ where $f(n) \in \Theta(n^k)$

$a < b^k$	$T(n) \in \Theta(n^k)$
$a = b^k$	$T(n) \in \Theta(n^k \log n)$
$a > b^k$	$T(n) \in \Theta(n^{\log_b a})$

a=2, b=2, k=0
$a>b^k$, C(n) belongs to Θ (n)