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Note Title 10/14/2009

The propositional calculus

Syntax: • propositional variables, $p_1, p_2, \ldots$ as needed
• parentheses, (,)
• connectives: $\sim$ (not)
$\supset$ (implies)

• formulas: (1) variables are formulas
(2) if f is a formula, $\sim f$ is a formula
(3) if f and g are formulas, $(f \supset g)$ is a formula

Logical axioms.

For every formula, A, B, C,

[actually, axiom schemata!]

$(A \supset (B \supset A))$ is an axiom (axiom 1)

$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)))$ is an axiom (axiom 2)

$((\sim A \supset \sim B)) \supset (B \supset A))$ " " " (axiom 3)

Rule of inference

$\{A, (A \supset B)\} \rightarrow B$ modus ponens

(Read: from A and $(A \supset B)$, conclude B

(The Hilbert style formulation of the propositional calculus)

From the (logical) axioms, using the rule of inference, you can <u>derive</u> formulas, which are called <u>theorems</u>.

Example:

$$\Big(\big(\overbrace{A}^{\text{"A" above}} \supset (\overbrace{(B \supset A)}^{\text{"B" above}} \supset \overset{C}{A})\big) \supset \big((A \supset (B \supset A)) \supset (A \supset \overset{C}{A})\big)\Big)$$ axiom 2

$$\Big[\ (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))\ \Big]$$ see!

$(A \supset ((B \supset A) \supset A))$ axiom 1 (with ~~A ← A~~, ~~B~~ (B ⊃ A))

$((A \supset (B \supset A)) \supset (A \supset A))$ m.p. on first 2 steps

$(A \supset (B \supset A))$ axiom 1

$(A \supset A)$ ✓ m.p. on latest 2 steps

Fairly complicated!

To indicate that a formula, say A is a theorem of the propositional calculus, we write

$$\vdash A.$$

(This means there is a derivation of A ....)

<u>Defn.</u> Proof from hypotheses.

F is derivable from hypotheses $\mathcal{H} = \{H_1, \ldots, H_n\}$ if there is a derivation of F using modus ponens, the logical axioms, and the hypotheses.

Hypotheses are sometimes called non-logical axioms.
If F can be derived from hypotheses ~~H~~ H, you write $H \vdash F$.
Trivial proof example:

$$A \vdash A$$

Deduction theorem

If $B_1, B_2, \ldots, B_{k-1}, B_k \vdash C$, then

$$B_1, B_2, \ldots B_{k-1} \vdash (B_k \supset C)$$

(The converse is also true.)

Semantics

Truth tables provide the truth value of $\wedge$, $\supset$.
The truth value of other formulas is built according to their structure, in the obvious way.
A formula is valid (or: a tautology) if it is true in every row of its truth table (i.e., it is true for every assignment of true or false to its propositional variables).

Contradiction

Satisfiable formula

If F is a tautology, you write $\models F$.

Theorem 1: If $\vdash F$, then $\models F$.

(If there is a derivation of F, i.e., F is a theorem, i.e, F can be proved, then F is a tautology) (Soundness of the p.c)

Theorem 2. If $\models F$ then $\vdash F$.

(Completeness of the P.C.)

(Theorem 1 is easy.)

(Theorem 2 is harder.)

There are other proof techniques than the Hilbert-style one.

E.g., natural deduction

(See the presentation on it on the web site.)