317 Ch. 9 [H] Ergodicity Theory

Note Title 2015-04-14

$\pi_j = \lim_{n\to\infty} P_{ij}^n$ is an ensemble average.

1. When does the limiting probability exist?

2. How does π_j compare with the long-run time average spent in state j, p_j?

3. What can we say about the mean time between visits to state j, and how is this related to π_j?

9.2 Finite-State DTMCs

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad P^2 = P \cdot P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P^3 = P^2 \cdot P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = P$$

So, $\quad P^{2n} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, but $P^{2n-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $n \geq 1$

and $\lim_{n \to \infty} P^n$ does not exist.

However, the time average spent in each state is the same, so

$p_0 = p_1 = \frac{1}{2}$

Also, the stationary distribution exists;

$$\vec{\pi} P = \vec{\pi} \iff (\pi_0 \quad \pi_1)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = (\pi_0 \quad \pi_1)$$

$$\begin{cases} \pi_1 = \pi_0 \\ \pi_0 = \pi_1 \\ \pi_0 + \pi_1 = 1 \end{cases} \Rightarrow \quad \pi_0 = \pi_1 = \frac{1}{2} \left(= p_1 = p_2\right)$$

Question: Does the following transition matrix have limiting probabilities?

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

4 states: 0, 1, 2, 3

Is this periodic? (If so, it does not have limiting probabilities.)

(1) Starting from state 0, you can get to states 2 and 3 (only) in one step

(2) From states 2 & 3, you get to state 1 (only) in one (more) step.

(3) From state 1, you get to state 0 (only) in one more step.

The same is true of the other states: from state 1, you get to ϕ in 1 step, then, (1), and (2) bring you back to state 1.

Similarly from states 2 and 3.

So, all states have period 3.

So, the chain for which P above is the transition matrix is periodic with period 3.

So, there is no limiting frequency.

Definition 9.1 The ***period*** of state j is the greatest common divisor (GCD) of the set of integers n, such that $P_{jj}^n > 0$. A state is ***aperiodic*** if it has period 1. A chain is said to be aperiodic if all of its states are aperiodic.

Definition 9.2 State j is ***accessible*** from state i if $P_{ij}^n > 0$ for some $n > 0$. States i and j ***communicate*** if i is accessible from j and vice versa.

Definition 9.3 A Markov chain is ***irreducible*** if all its states communicate with each other.

Question: Do you think that aperiodicity and irreducibility are enough to guarantee the existence of the limiting distribution?

Answer: As we see in Theorem 9.4, for a finite-state DTMC, aperiodicity and irreducibility are all that is needed to ensure that the limiting probabilities exist, are positive, sum to 1, and are independent of the starting state. This is very convenient, because it is often easy to argue that a DTMC is aperiodic and irreducible.

Theorem 9.4 *Given an aperiodic, irreducible, finite-state DTMC with transition matrix* $\mathbf{P}$, *as* $n \to \infty$, $\mathbf{P}^n \to \mathbf{L}$, *where* $\mathbf{L}$ *is a limiting matrix all of whose rows are the same vector,* $\vec{\pi}$. *The vector* $\vec{\pi}$ *has all positive components, summing to* 1.

9.2.2. Mean time between Visits to a State

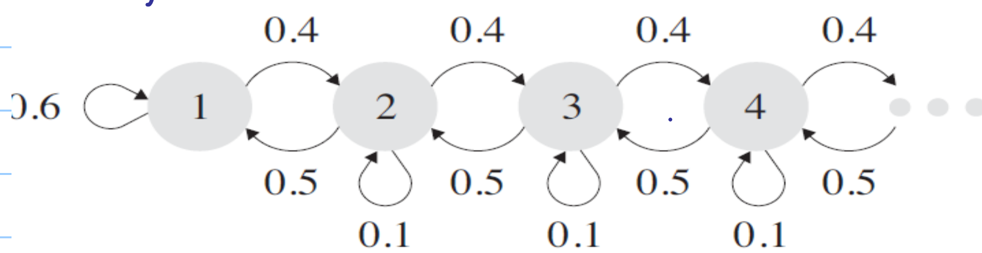
Definition 9.5 Let m_{ij} denote the expected number of time steps needed to first get to state j, given we are currently at state i. Likewise, let m_{jj} denote the expected number of steps between visits to state j.

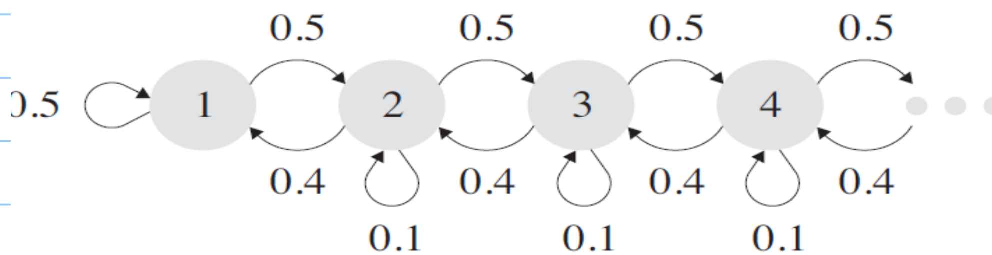
Theorem 9.6 *For an irreducible, aperiodic finite-state Markov chain with transition matrix* $\mathbf{P}$,

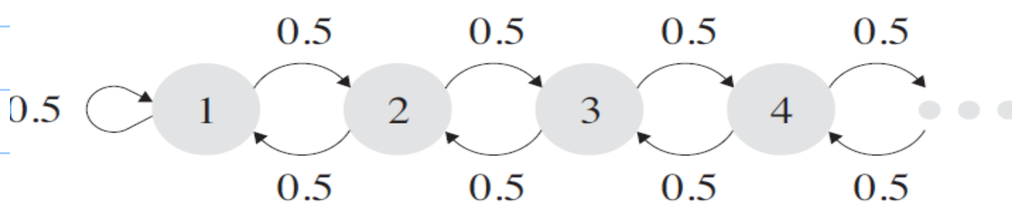
$$m_{jj} = \frac{1}{\pi_j}$$

where m_{jj} is the mean time between visits to state j and $\pi_j = \lim_{n\to\infty} (\mathbf{P}^n)_{ij}$.

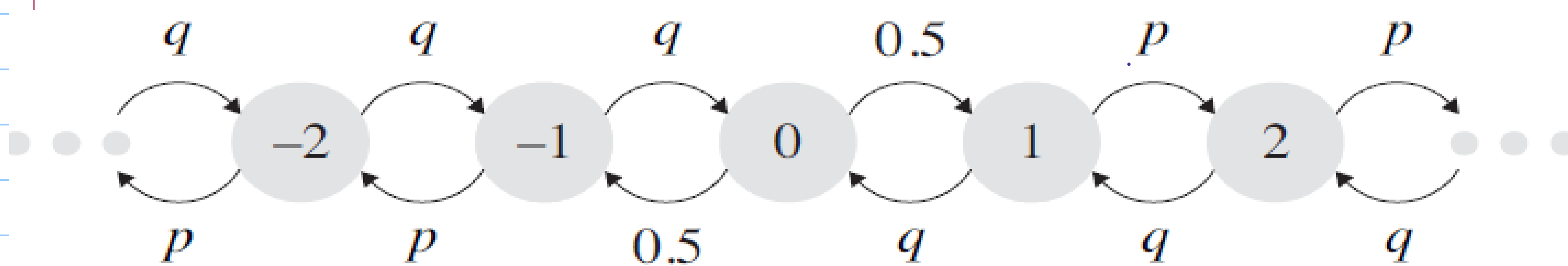
9.3 Infinite-State Markov Chains







9.3.2 Infinite Random Walk Example



Theorem 9.27 (Summary Theorem) *An irreducible, aperiodic DTMC belongs to* one *of the following two classes:*

Either

(i) *All the states are transient, or all are null recurrent. In this case* $\pi_j = \lim_{n\to\infty} P_{ij}^n = 0, \forall j$, *and there does NOT exist a stationary distribution.*

or

(ii) *All states are positive recurrent. Then the limiting distribution* $\vec{\pi} = (\pi_0, \pi_1, \pi_2, \ldots)$ *exists, and there is a positive probability of being in each state. Here*

$$\pi_j = \lim_{n\to\infty} P_{ij}^n > 0, \quad \forall i$$

is the limiting probability of being in state j. *In this case* $\vec{\pi}$ *is a stationary distribution, and no other stationary distribution exists. Also,* π_j *is equal to* $\frac{1}{m_{jj}}$, *where* m_{jj} *is the mean number of steps between visits to state* j.

Theorem 9.34 (Time-reversible DTMC) *Given an aperiodic, irreducible Markov chain, if there exist* $x_1, x_2, x_3, \ldots$ *s.t.,* $\forall i, j$,

$$\sum_i x_i = 1 \quad \textit{and} \quad x_i P_{ij} = x_j P_{ji},$$

then

1. $\pi_i = x_i$ *(the* x_i*'s are the limiting probabilities).*
2. *We say that the Markov chain is time-reversible.*

Example: Three Types of Equations

Consider the Markov chain depicted in Figure 9.5.

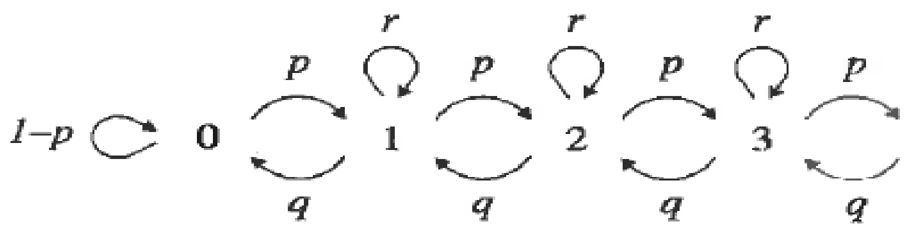


Figure 9.5. A very familiar Markov chain.

Regular Stationary Equations:

$$\pi_i = \pi_{i-1}p + \pi_i r + \pi_{i+1}q \quad \text{and} \quad \sum_i \pi_i = 1$$

These are messy to solve.

Balance Equations:

$$\pi_i(1-r) = \pi_{i-1}p + \pi_{i+1}q \quad \text{and} \quad \sum_i \pi_i = 1$$

These are a little nicer, because we are ignoring self-loops, but still messy to solve.

Time-Reversibility Equations:

$$\pi_i p = \pi_{i+1}q \quad \text{and} \quad \sum_i \pi_i = 1$$

These are *much simpler* to solve.