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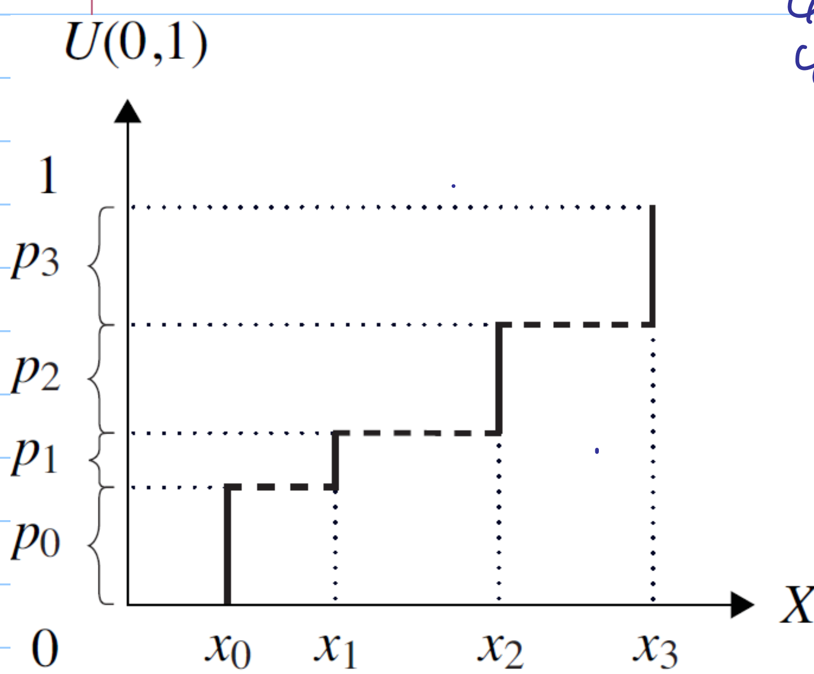
Note Title 2015-02-19

Generating r.v.s for simulation (Ch. 4)



4.1 Inverse-Transform Method

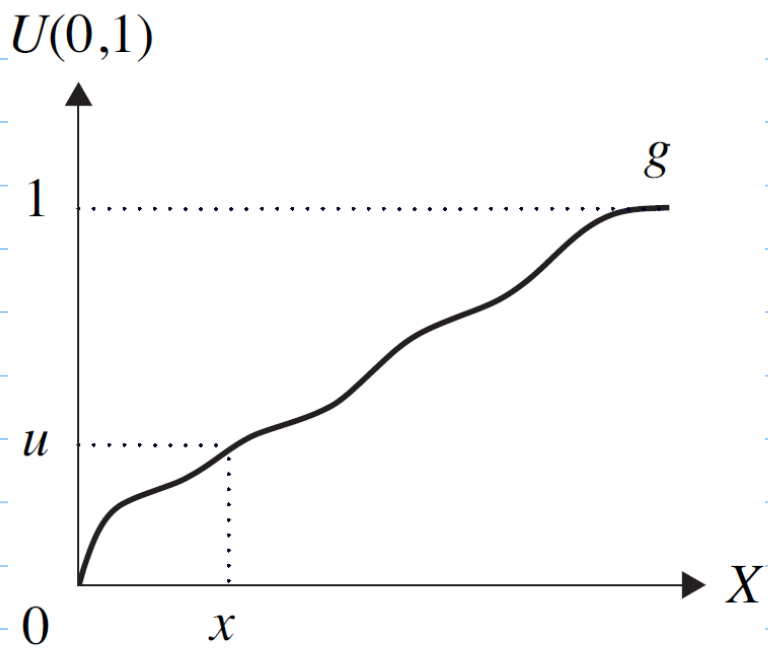
4.1.2 Discrete case

$$X = \begin{cases} x_0 & \text{with prob } p_0 \\ x_1 & \text{" " } p_1 \\ \vdots & \\ x_k & \text{" " } p_k \end{cases}$$

1. Arrange $x_0 \dots x_k$ in order
2. Generate $u \in U(0,1)$
3. (If $0 < u \le p_1$ then output x_0)

   If $\sum_{i=0}^{\ell-1} p_i < u \le \sum_{i=0}^{\ell} p_i$ then output x_ℓ

4.2.1 The continuous case



A value in $(0,x)$ should be output with probability $F_X(x)$

We want $\doteq P(0<U<u) = P(0<X<x) =$

But $u \overset{!}{=}$

$F_X(x) \overset{!}{=}$

and

(where $F_X(\cdot)$ is the cdf of rv X)

So, we want $u = F_X(x)$ or equivalently $x = F^{-1}(u)$

This gives the Inverse-Transform method to generate r.v. X,

1. Generate $u \in U(0,1)$
2. Return $x = F_X^{-1}(u)$

Example: Generate $X \sim Exp(\lambda)$

$F(x) = 1 - e^{-\lambda x}$

So, we want $x = F^{-1}(u) \Rightarrow F(x) = u \Rightarrow 1 - e^{-\lambda x} = u \Rightarrow$

$$\Rightarrow -\lambda x = \ln(1-u) \Rightarrow x = -\frac{1}{\lambda}\ln(1-u)$$

Given $u \in U(0,1)$, setting $x = -\frac{1}{\lambda}\ln(1-u)$ produces an instance of $X \sim Exp(\lambda)$.

The above result can be obtained as an application of the following Theorem [3.1, Trivedi]. Let X be a continuous r.v. with density f_X which is nonzero on a subset I of real numbers (i.e, $f_X(x) > 0$, $x \in I$ and $f_X(x) = 0$, $x \notin I$).
Let Φ be a differentiable monotone (invertible) function whose domain is I and whose range is the set of reals.
Then, $Y = \Phi(X)$ is a continuous r.v. with the density f_Y given by

$$f_Y(y) = \begin{cases} f_X\left[\Phi^{-1}(y)\right]\left[\left|(\Phi^{-1})'(y)\right|\right], & y \in \Phi(I) \\ 0, & \text{otherwise} \end{cases},$$

where Φ^{-1} is the uniquely defined inverse of Φ and $(\Phi^{-1})'$ is the derivative of the inverse function.

Proof (omitted)

Example.

Let Φ be the distribution function, F, of an r.v. X

with density f. Applying the above theorem, $Y = F(X)$ and $F_Y(y) = F_X\left(F_X^{-1}(y)\right) = y$. Therefore, the r.v. $Y = F(X)$ has the density given by

$$f_Y = \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

In other words, if X is a continuous r.v with CDF F, then the new r.v. $Y = F(X)$ is uniformly distributed over the interval $(0,1)$.

4.2 Accept-Reject Method

4.2.1 Discrete case

Let c be a constant s.t. $\frac{p_j}{q_j} \leq c$ $\forall j$ s.t. $p_j > 0$

1. Find r.v. Q s.t. $q_j > 0 \iff p_j > 0$
2. Generate an instance of Q, and call it j.
3. Generate instance u of r.v. $U(0,1)$
4. If $u < \frac{p_j}{c q_j}$, return $P = j$ and stop; otherwise, return to step 2.

Here is a proof of correctness. We want to show that

$P\{P$ ends up being set to j (as opposed to some other value)$\} = p_j$.

$$P\{P \text{ ends up being set to } j\} = \frac{\text{Fraction of time } j \text{ is generated \& accepted} \; ①}{\text{Fraction of time any value is accepted} \; ②}$$

$$① = P\{j \text{ is generated}\} \cdot P\{j \text{ is accepted} \mid j \text{ is generated}\} =$$

$$= q_j \cdot \frac{p_j}{c q_j} = \frac{p_j}{c}$$

$$② = \sum_j \text{fraction of time } j \text{ is generated \& accepted} = \sum_j \frac{p_j}{c} = \frac{1}{c}.$$

So, $P\{P \text{ ends up being set to } j\} = \dfrac{\frac{p_j}{c}}{\frac{1}{c}} = p_j$ ✓

Note that, from (2), we can derive that c values are generated on average before one is accepted.

4.2.2. Continuous case (of Accept-Reject method)

1. Find continuous r.v. Y s.t. $f_Y(t)>0 \Leftrightarrow f_X(t)>0$.

Let c be a constant s.t.

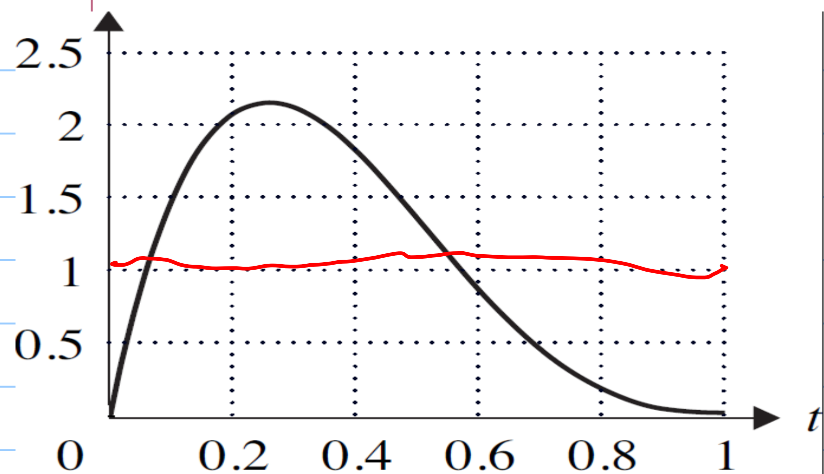
$$\frac{f_X(t)}{f_Y(t)} \leq c \quad \forall t \text{ s.t. } f_X(t)>0$$

2. Generate an instance t of Y.

3. With prob. $\dfrac{f_X(t)}{c \cdot f_Y(t)}$, return $X=t$ (i.e., "accept t" and stop).
Else reject t and return to step 2

Question. How does one return "with prob p"?

Answer: as in the discrete case, i.e.: generate an instance, u, of $U(0,1)$ (using a random number generator). If $u \le p$ accept. If $u > p$, reject.



We want to generate an r.v. X with pdf $f_X(t) = 20t(1-t)^3$, $0 < t < 1$

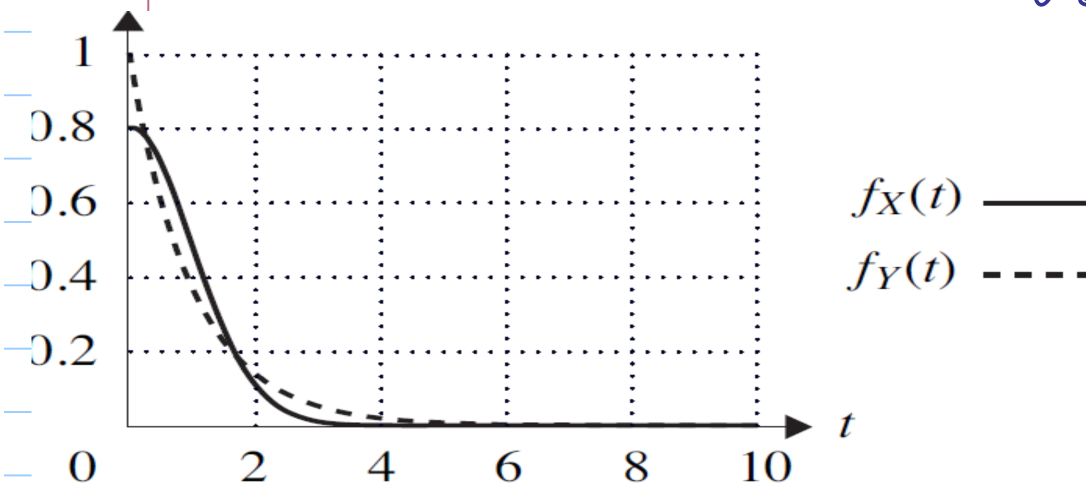
$Y \sim U(0,1)$

$$\frac{f_X(t)}{f_Y(t)} = f_X(t) \leq \frac{135}{64}$$

max $f_X(t)$ in $0 \leq t \leq 1$

Generate $N \sim \text{Normal}(0,1)$.

We will instead generate $X = |N|$ and multiply by -1 with prob. $\frac{1}{2}$.



$$f_N(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}, \quad -\infty < t < \infty$$

$$f_X(t) = f_{|N|}(t) = 2 f_X(t) = \frac{2}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$$

Choose $Y \sim \text{Exp}(1)$, so $f_Y(t) = e^{-t}$, $0 < t < \infty$

$$g(t) = \frac{f_X(t)}{f_Y(t)} = \frac{2}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \Big/ e^{-t} = \frac{2}{\sqrt{2\pi}} e^{-\frac{t^2}{2}+t} = \sqrt{\frac{2}{\pi}}\, e^{t-\frac{t^2}{2}} \quad (*)$$

max $g(t)$ occurs for max $\left(t - \frac{t^2}{2}\right)$

$\frac{d}{dt}\left(t - \frac{t^2}{2}\right) = 1 - t \Rightarrow t = 1$ maximizes $g(t)$

The corresponding value of $g(t) = \frac{f_x(1)}{f_y(1)} = \sqrt{\frac{2e}{\pi}} \approx 1.3$

$g(1)$

$~~f~~(1) = \sqrt{\frac{2}{\pi}}\, e^{1-\frac{1}{2}} = \sqrt{\frac{2}{\pi}}\, e^{\frac{1}{2}} = \sqrt{\frac{2e}{\pi}}$