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Thm 3.25

For discrete r.v.s; $E[X] = \sum_y E[X \mid Y=y] \, P\{Y=y\}$ (*)

For continuous r.v.s, $E[X] = \int_{-\infty}^{\infty} E[X \mid Y=y] \, f_Y(y) \, dy$

Based on the law of total probability

$$P\{X=k\} = \sum_y P\{X=k \mid Y=y\} \, P\{Y=y\}$$

This allows "case analysis"

We use (*) to compute the mean of $X \sim \text{Geometric}(p)$

We condition on Y, the value of the first flip

$$E[X] = \underbrace{E[X \mid Y=1]}_{1}\,\underbrace{P\{Y=1\}}_{p} + E[X \mid Y=0]\,\underbrace{P\{Y=0\}}_{1-p} =$$

$$= 1 \cdot p + (1 + E[X])(1-p)$$

$$\cancel{E[X]} = \cancel{p} + (1 - \cancel{p}) + \cancel{E[X]} - p\,E[X]$$

$$E[X] = \frac{1}{p}$$

Recall

Thm 3.26 (Linearity of Expectation)

$E[X+Y] = E[X] + E[Y]$ for any two r.v.s X and Y.

Example: Binomial (n, p).

Let $X \sim \text{Binomial}(n, p)$

$$E[X] = \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i}$$ Scary! But we can think

of X as a sum of n indep. & identically distributed r.v.s,

which we call X_i, $1 \le i \le n$,

$$X_i = \begin{cases} 1 & \text{if flip } i \text{ is successful} \\ 0 & \text{otherwise} \end{cases}$$

$$E[X_i] = p$$

$$E[X] = E[X_1] + \dots E[X_n] = \underbrace{p + \dots + p}_{n \text{ times}} = np$$

X_i are <u>indicator</u> random variables.

Example: Hats

.... game .... hats.... n players .... n hats ...

The r.v. X is the number of players who get back their own hats.

$$I_i = \begin{cases} 1 & \text{if person } i \text{ gets back his/her own hat} \\ 0 & \text{otherwise} \end{cases}$$

$$X = I_1 + I_2 + \cdots + I_n$$

Note that the I_i's are <u>not</u> independent r.v.s. However, linearity of expectation still holds, so

$$E[X] = E[I_1] + E[I_2] + \dots + E[I_n] = n\,E[I_i] =$$
$$n\left(\sum_{i=0}^{1} i\,p_I(i)\right) = n\left(0 \cdot \frac{n-1}{n} + 1 \cdot \frac{1}{n}\right) = n \cdot \frac{1}{n} = 1$$