# Globally Optimal Nonlinear Model Predictive Control

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**Presentation Outline**

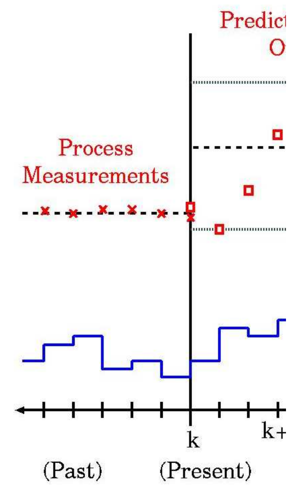
- Background and Motivation
- Nonlinear Model Predictive Control
  - Formulation
  - Deterministic Method for Solving Nonconvex Problem
- Case Study
  - Isothermal CSTR with Van de Vusse kinetics
- Summary

# Model Predictive Control (MPC)

## (Garcia, Prett, and Morari, 1989)

- Model Predictive Control is an advanced control algorithm that handles:
  - Multivariable interacting systems
  - Soft constraints on both inputs and outputs
  - Measured disturbances
  - Process time delays and difficult dynamics
  - Known future setpoint changes (reference transitions)
- Limitations
  - Requires an explicit model of the process
  - Need to solve a constrained optimization problem online

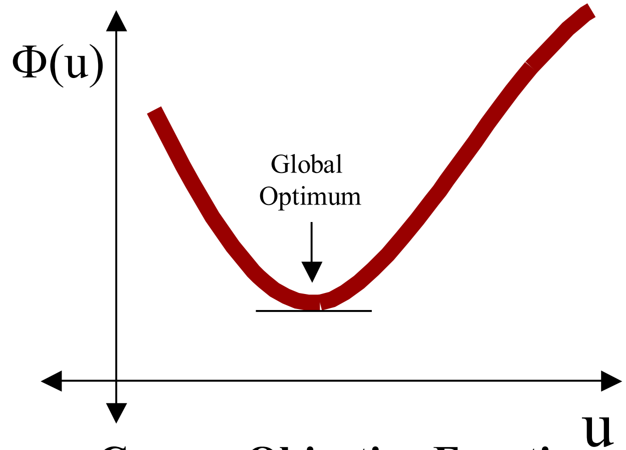
1. Using process data, formulates an optimization problem.
2. Solves the problem online and implements the optimal control move.
3. Waits for new data at next time step.



# Linear vs. Nonlinear Formulations

**Linear Formulations:**

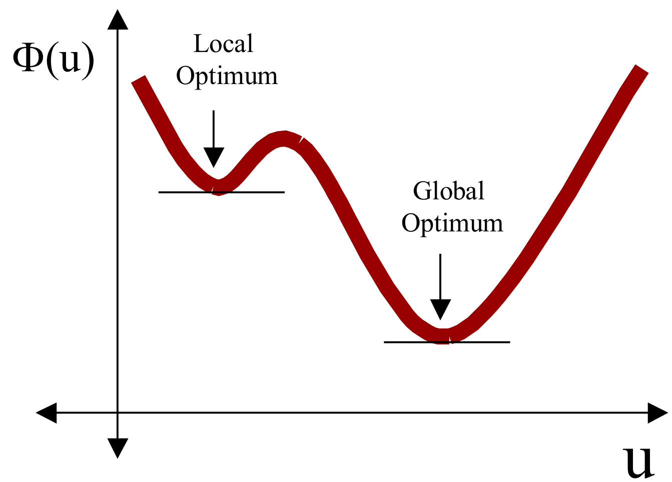
- Mismatch between linear model and nonlinear process can lead to poor closed-loop performance and/or instability.
- Linear model (linear constraints) → convex problem (only one minimum)
- Efficient solvers (local) exist to solve the convex problem




**Convex Objective Function**

**Nonlinear Formulations:**

- Nonlinear model handles process nonlinearities.
- Nonlinear model (nonlinear constraints) → nonconvex problem (multiple minima)
- Local solvers left susceptible to choosing suboptimal minima.
- Need to guarantee global optimality in nonconvex problems.



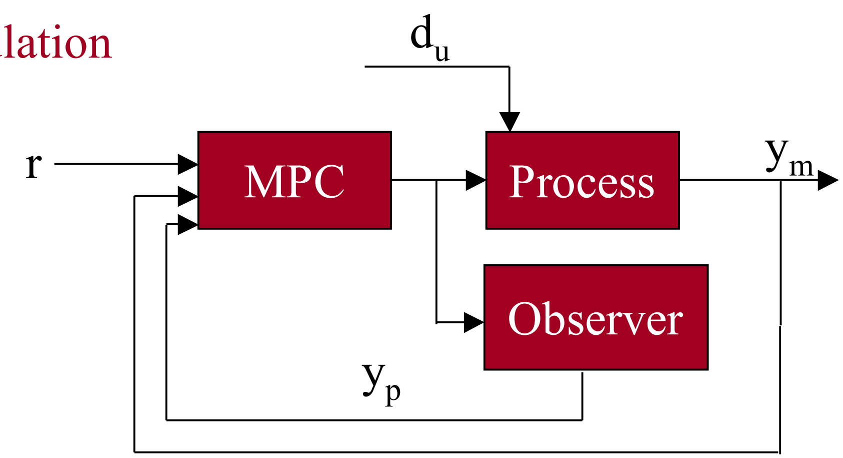

**Nonconvex Objective Function**

# Nonlinear Model Predictive Control (NMPC) Formulation

Objective Function:

1-norm formulation

$$\Phi = \sum_{i=1}^{p} \Gamma_e e(i) + \sum_{i=1}^{m} \Gamma_u \Delta u(i)$$




Constrained Optimization Problem:

$$\min_{\{u(i), \ldots, u(m)\}} \Phi$$

subject to:

$$x(k+1) = f(x(k), u(k))$$
$$y(k) = g(x(k), u(k))$$

**Nonlinear Process Model**

$$|r(i) - y(i)| \le e(i) \quad \forall i = 1 \ldots p$$ **Error Terms**

$$|u(i-2) - u(i-1)| \le \Delta u(i-1) \quad \forall i = 1 \ldots m$$ **Δu terms**

$$d(i) = y_m(0) - y_p(0)$$ **Disturbance Update**

$$u^L \le u \le u^U$$ **Actuator Limits**

r = reference (setpoint)
$y_m$ = measured output
$d_u$ = unmeasured disturbance
$y_p$ = predicted output

Re-written in a more compact form

$$\min_x C^T x$$
$$s.t. \quad Ax \le b$$
$$f(x) = 0$$
$$x^L \le x \le x^U$$

Nonconvex NLP

# Online vs. Offline Methods for MPC

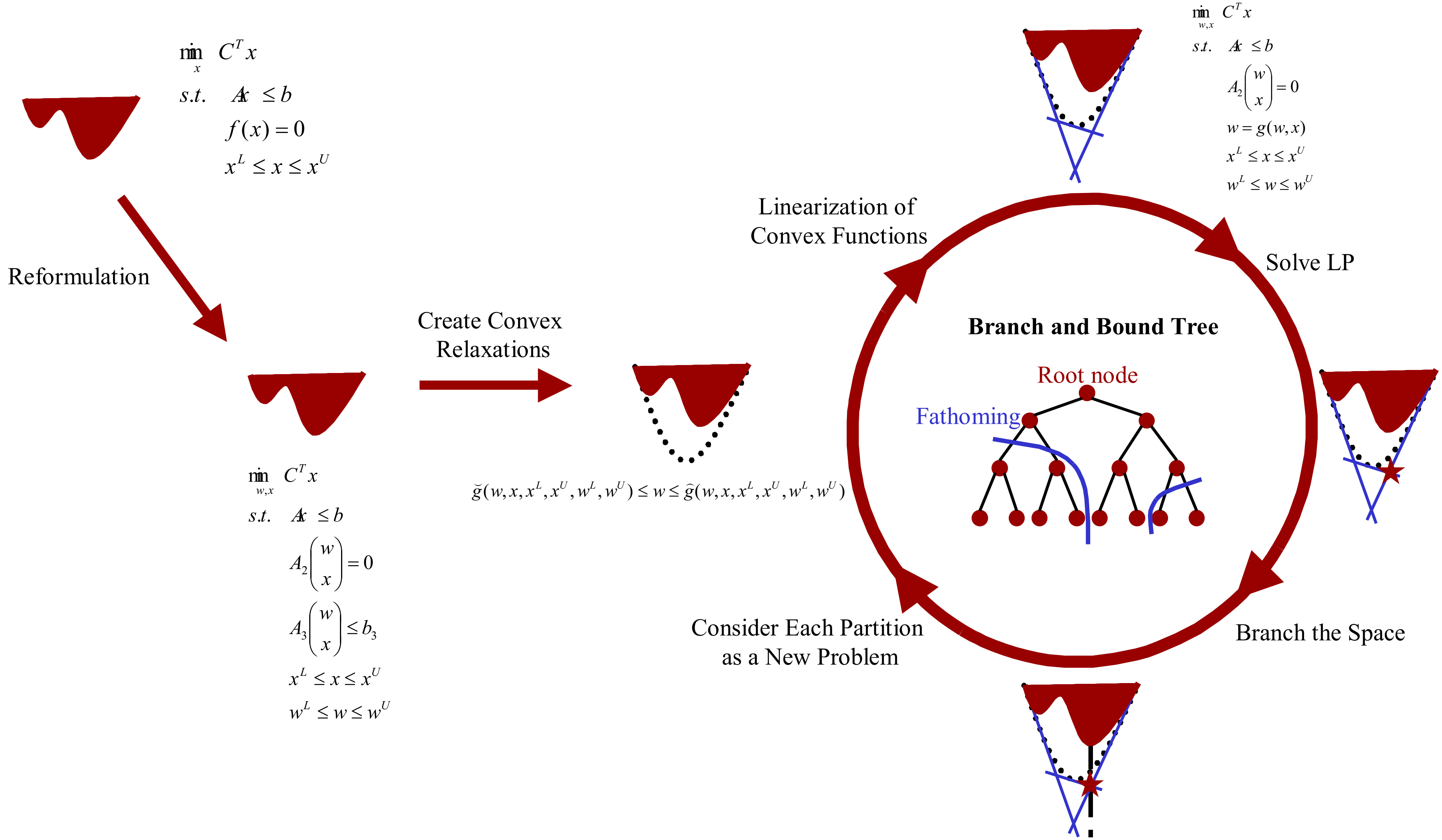
## Online

- Formulate the appropriate optimization problem based on current data at each time step.
- Solve the problem online.
  - computationally demanding
  - real-time constraints
- Possibility of unnecessarily solving the same problem over and over again
- Solution is optimal and pertains exactly to the process' current state.

## Offline

- Partition solution space into characteristic regions based on a set of parameters (states, inputs, etc…)
- Solve problems from each region offline
  - high dimensionality issues
  - Can you foresee all scenarios?
- At each time step, identify appropriate region online based on current data and "look up" the solution.
  - low online computational demand
- Implement solution from the region.
  - Suboptimal?

# Deterministic Method for Solving the Nonconvex NLP

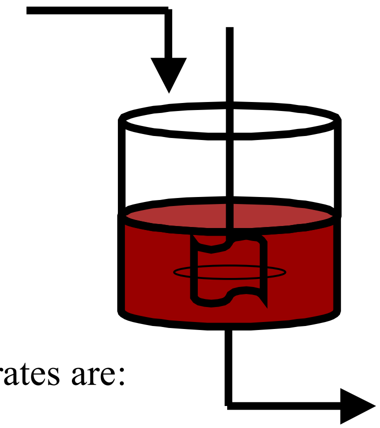


# Benchmark Control Problem

Consider the isothermal operation of a SISO two state CSTR exhibiting Van de Vusse kinetics:

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$

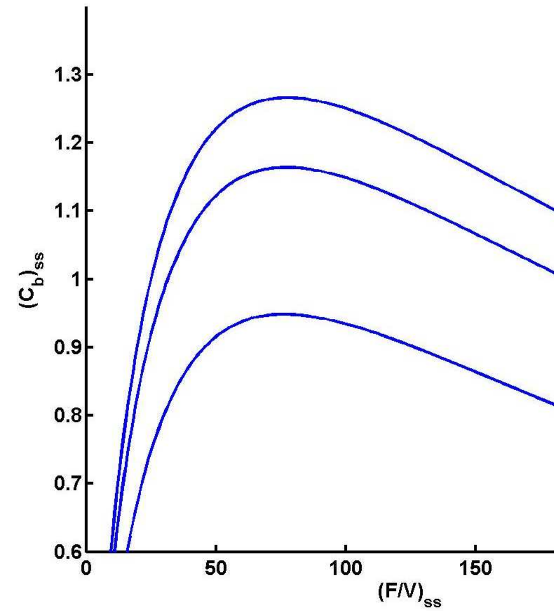
$$2A \xrightarrow{k_3} D$$



Where the corresponding reaction rates are:

$$r_A = -\,k_1 C_A - k_3 C_A^2$$

$$r_B = \; k_1 C_A - k_2 C_B$$




**Steady State Loci for the Reactor Operation at Different Feed Concentrations ($C_{AO}$) Exhibiting the Presence of an Input Multiplicity**

The system can be described by:

$$\frac{dC_A}{d} = \left(\frac{F}{V}\right)\left(C_{AO} - C_A\right) - k_1 C_A - k_3 C_A^2$$

$$\frac{dC_B}{d} = k_1 C_A - k_2 C_B - \left(\frac{F}{V}\right) C_B$$

where:

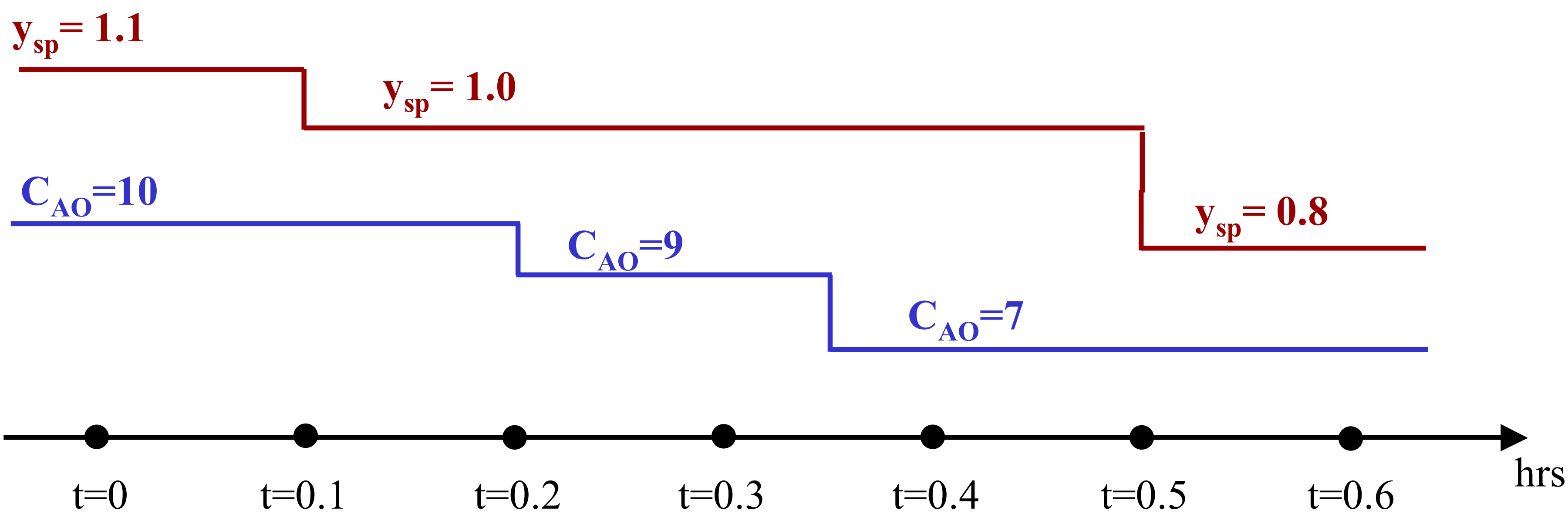
| | | | |
|---|---|---|---|
| $C_A$ | = | Conc. of Species A | = process state ($x_1$) |
| $C_B$ | = | Conc. of Species B | = process output ($y = x_2$) |
| $k_i$ | = | Reaction Rate Constants | |
| F | = | Feed Flow Rate | $\}$ (F/V) = process input (u) |
| V | = | Reactor Volume (constant) | |
| $C_{AO}$ | = | Conc. of A in the Feed | |

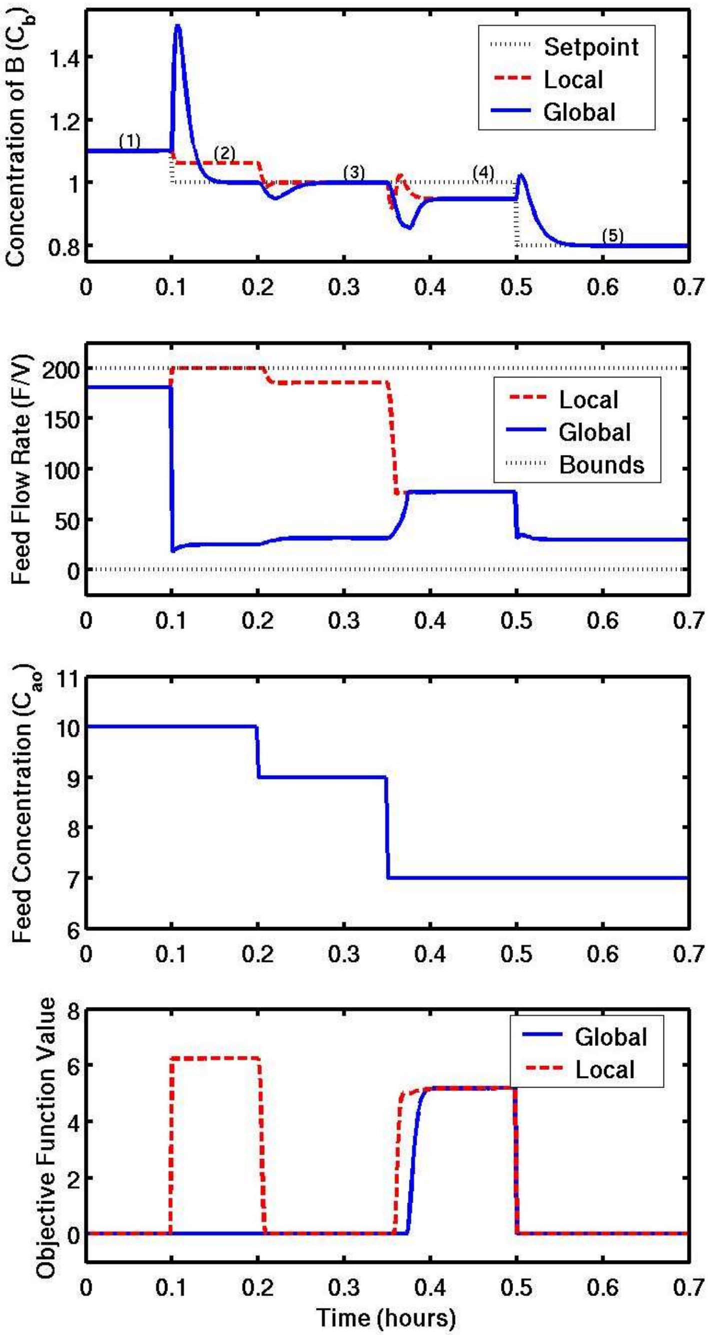
# Closed-loop Performance Test

Both setpoint tracking and disturbance rejection are tested through a series of setpoint transitions and disturbance loads.

Assume the process is initially being operating where:

- Feed Flow Rate/Reactor Volume (F/V) = 181 mol/liter-hr **process input (u)**
- Conc. of B ($C_B$) = 1.1 mol/liter **process output (y)**
- Conc. of A in the Feed ($C_{AO}$) = 10 mol/liter **disturbance (d)**






# Closed-loop Results

## (single degree of freedom)

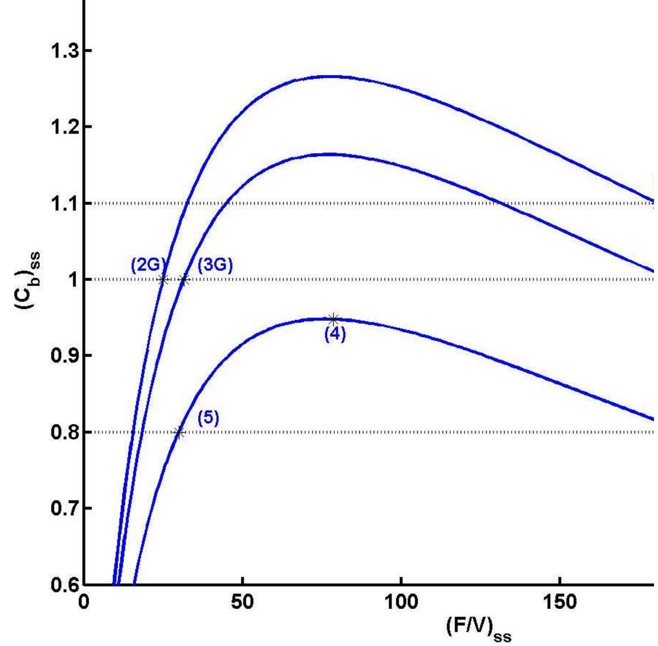
**Objective Function:**

$$\Phi = \Gamma_y e(p) + \sum_{i=1}^{m} \Gamma_u \Delta u(i)$$

**terminal weight**

**Tunings:** **m = 1** $\Gamma_y$=100

**p = 30** $\Gamma_u$=0




**Steady State Loci for the Reactor Operation at Different Feed Concentrations ($C_{AO}$) Exhibiting the Presence of an Input Multiplicity**

# Sample Optimization Problem

**Assume the process is operating at:**

**u = 181 mol/liter-hr**
**y = 1.1 mol/liter**
**$C_{AO}$ = 10 mol/liter**

**Consider the Objective Function as:**

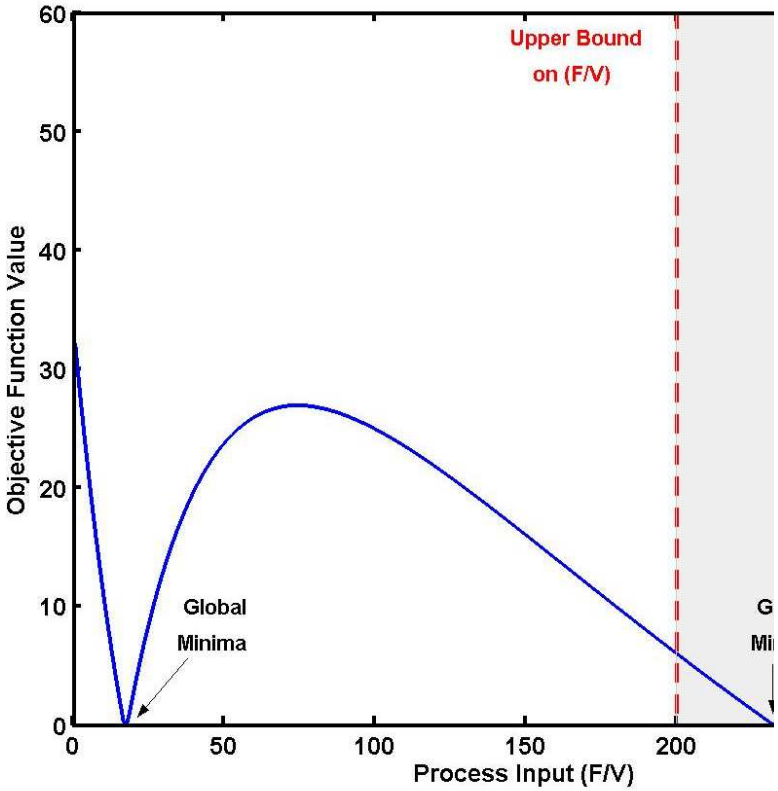
$$\Phi = \Gamma_y e(p) + \sum_{i=1}^{m} \Gamma_u \Delta u(i)$$

**(terminal weight)**

**Let: m = 1 $\Gamma_y$=100**
**p = 30 $\Gamma_u$=0**

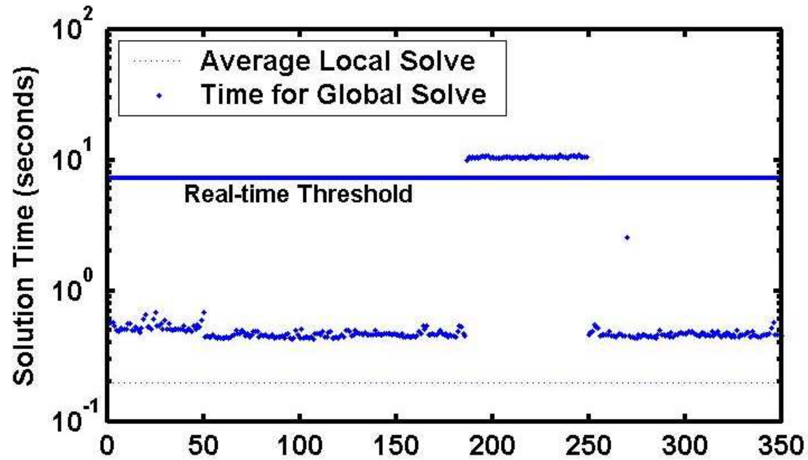
**Assume bounds on the input.**

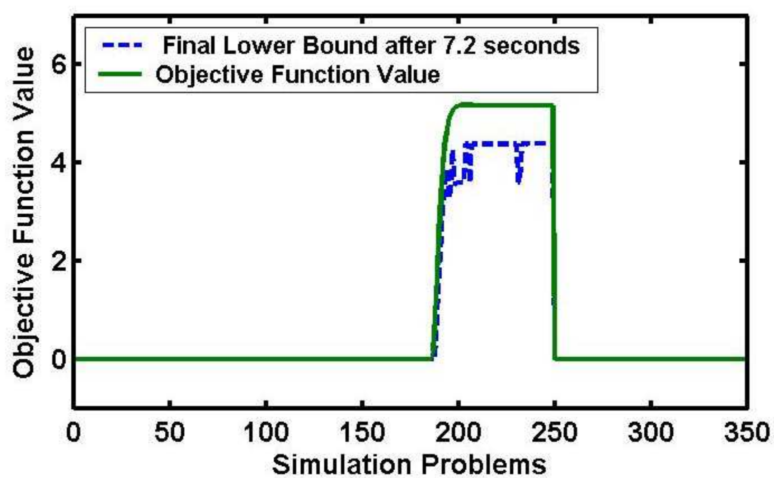
$$0 \leq u \leq 200$$



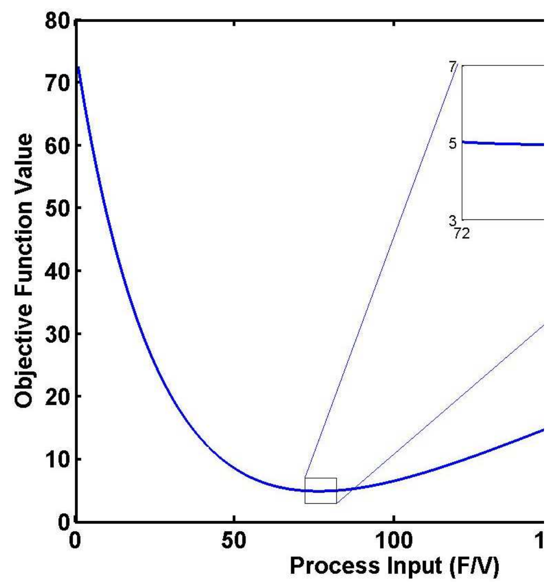

**Sample objective function for problem having the setpoint moved from the initial condition of 1.1 mol/liter to 1 mol/liter.**

# Real-Time Considerations



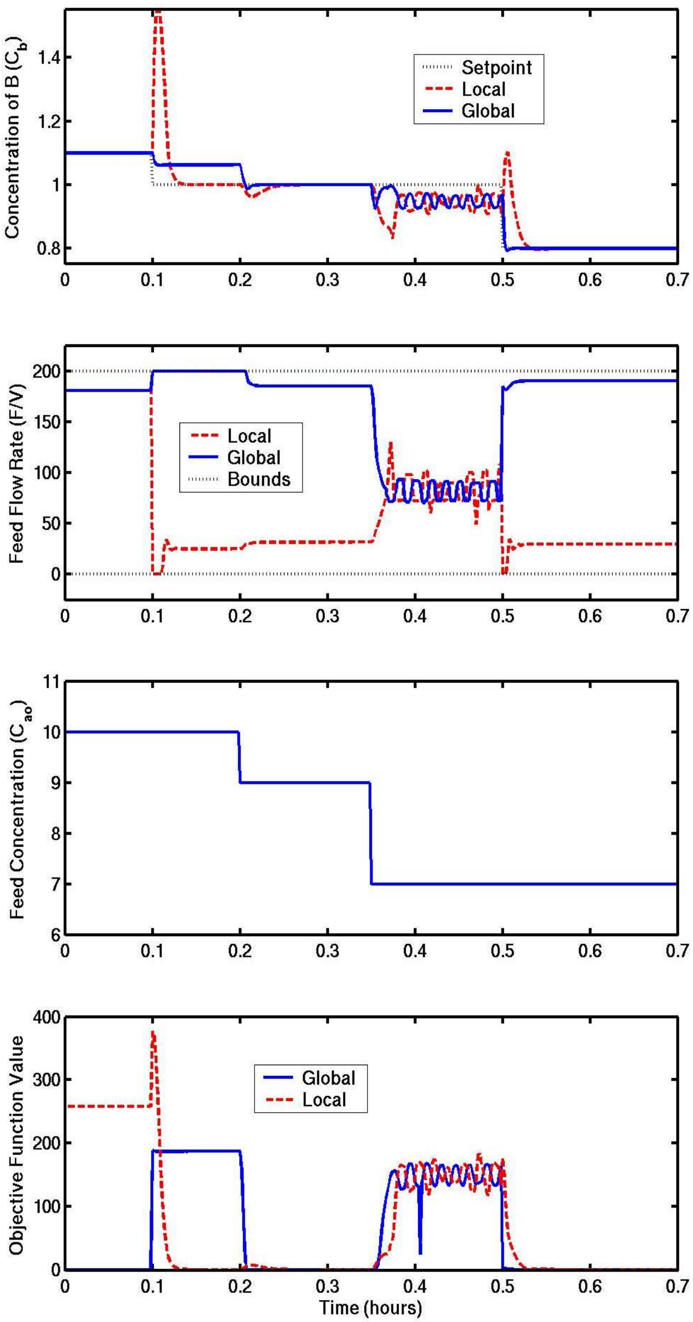




- Sampling Time = 7.2 seconds
- Must terminate solves to meet real-time constraints.
- Global solution might have been found, not guaranteed.




**Sample objective function for a problem that takes longer than 7.2 seconds to solve globally.**

# Closed-loop Results
## (multiple degrees of freedom)

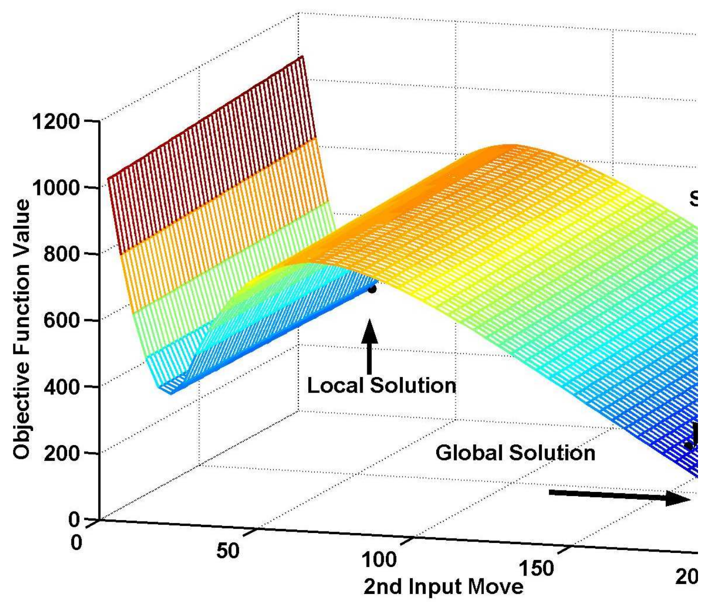



**Objective Function:**

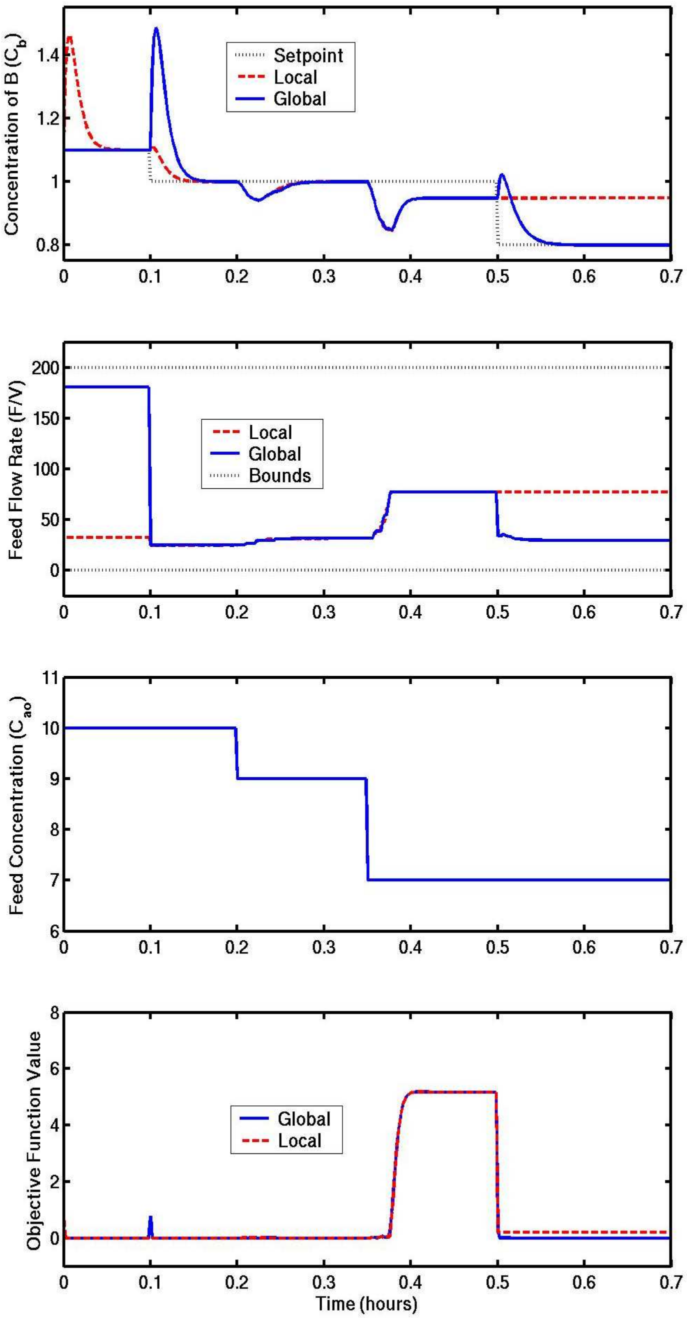
$$\Phi = \sum_{i=1}^{p} \Gamma_y e(i) + \sum_{i=1}^{m} \Gamma_u \Delta u(i)$$

**Tunings:** **m = 2** $\Gamma_y = 100$

**p = 30** $\Gamma_u = 0.005$




**Sample objective function at the time of the first setpoint change for the NMPC with two degrees of freedom (m=2)**




# Use of a Terminal Weight

## (multiple degrees of freedom)

**Objective Function:**

$$\Phi = \Gamma_y e(p) + \sum_{i=1}^{m} \Gamma_u \Delta u(i)$$

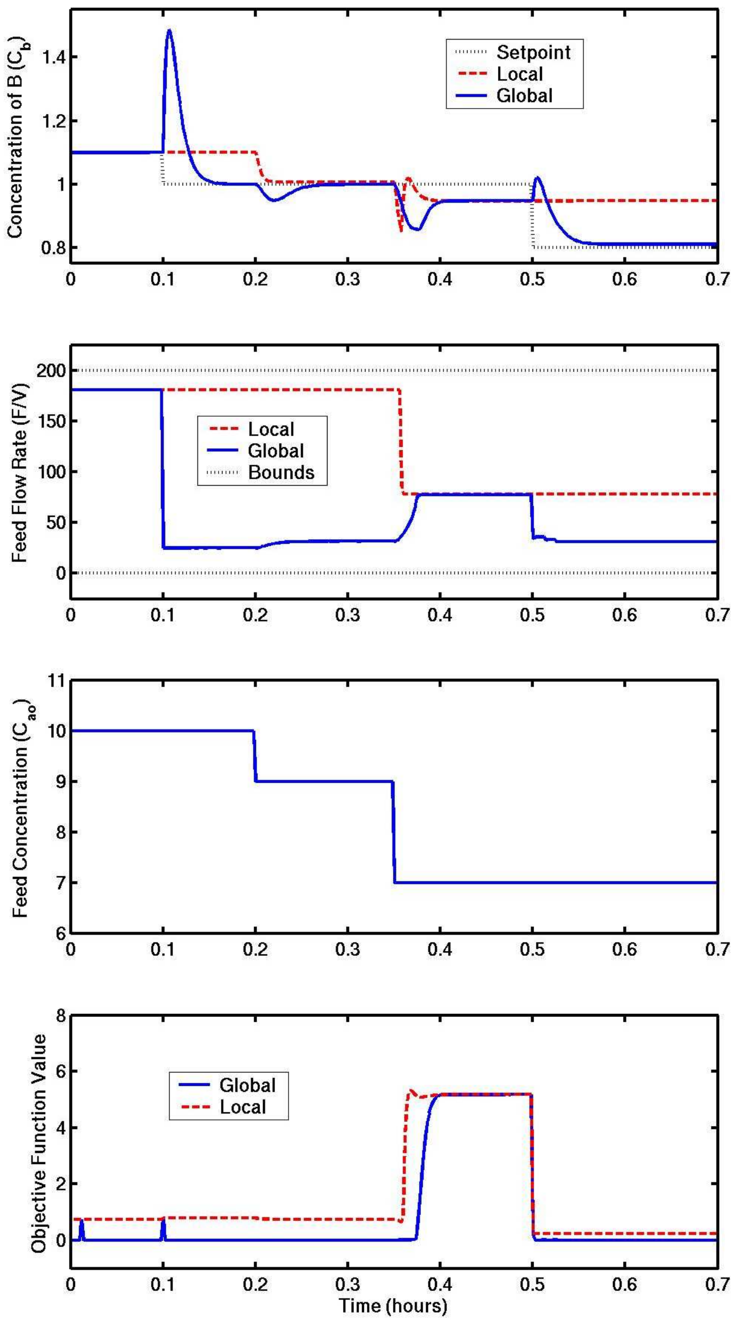
**Tunings:** **m = 2** **$\Gamma_y$=100**

**p = 30** **$\Gamma_u$=0.005**

**terminal weight**

**As in the single degree of freedom (m=1) case, the modified objective function (using the terminal weight) allows for the controller to better track the setpoint.**

# Incorporating Hard Constraints




**Objective Function:**

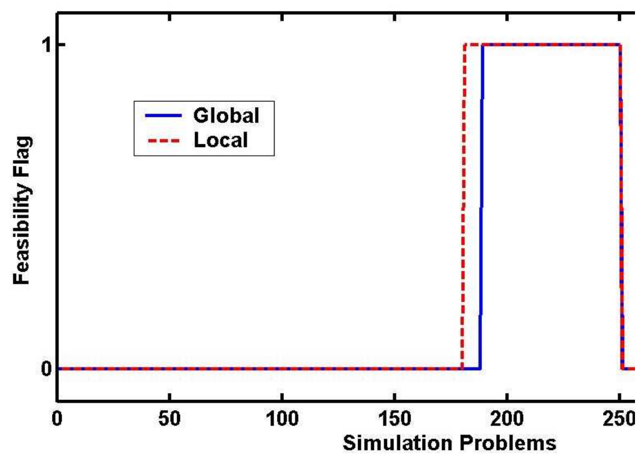
$$\Phi = \Gamma_y e(p) + \sum_{i=1}^{m} \Gamma_u \Delta u(i)$$

**Tunings:** **m = 2** **$\Gamma_y$=100**

**p = 30** **$\Gamma_u$=0.005**

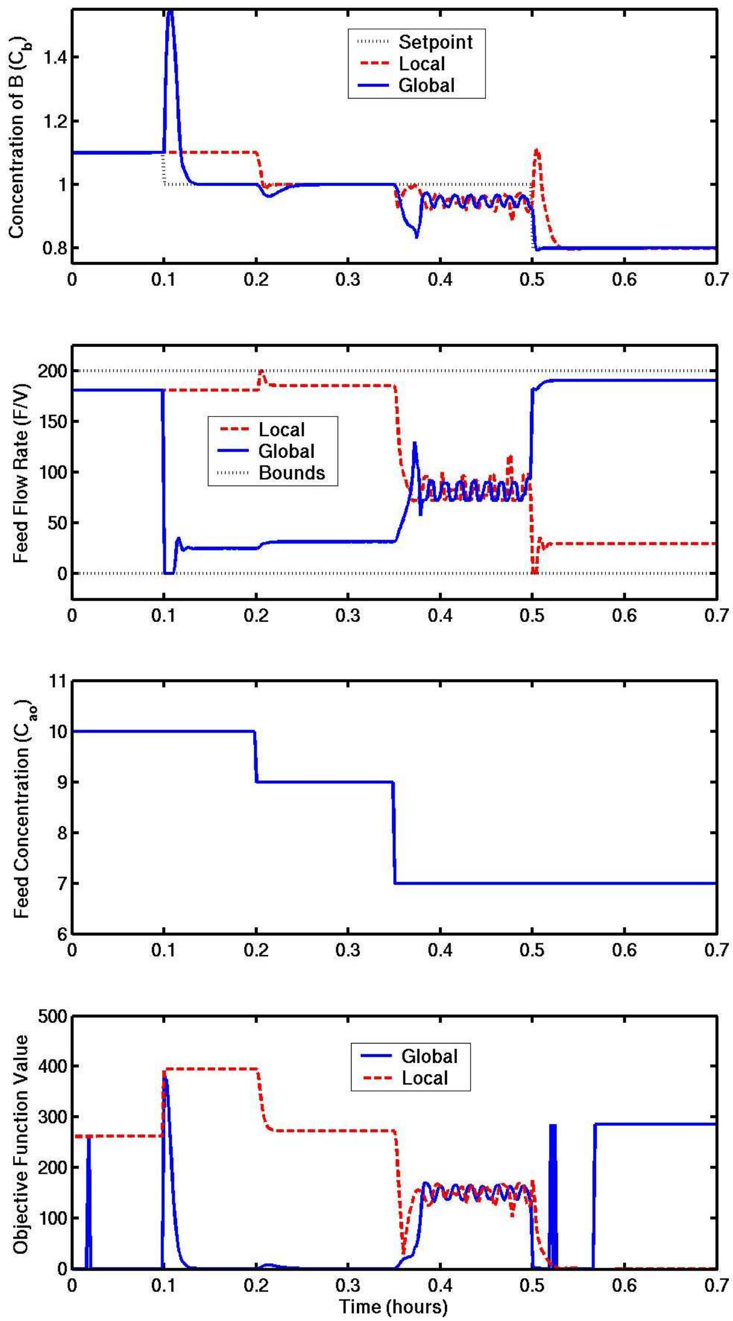
**e(p)=0**

**Possibility for Infeasible Problems**
**(constraint relaxation may be necessary!)**




**Problem feasibility when a hard constraint is imposed on e(p). A value of 0 indicates feasible problems, while a value of 1 shows infeasible problems in which a hard constraint relaxation is necessary.**

# Traditional Objective Function with Hard Constraint




**Objective Function:**

$$\Phi = \sum_{i=1}^{p} \Gamma_y e(i) + \sum_{i=1}^{m} \Gamma_u \Delta u(i)$$

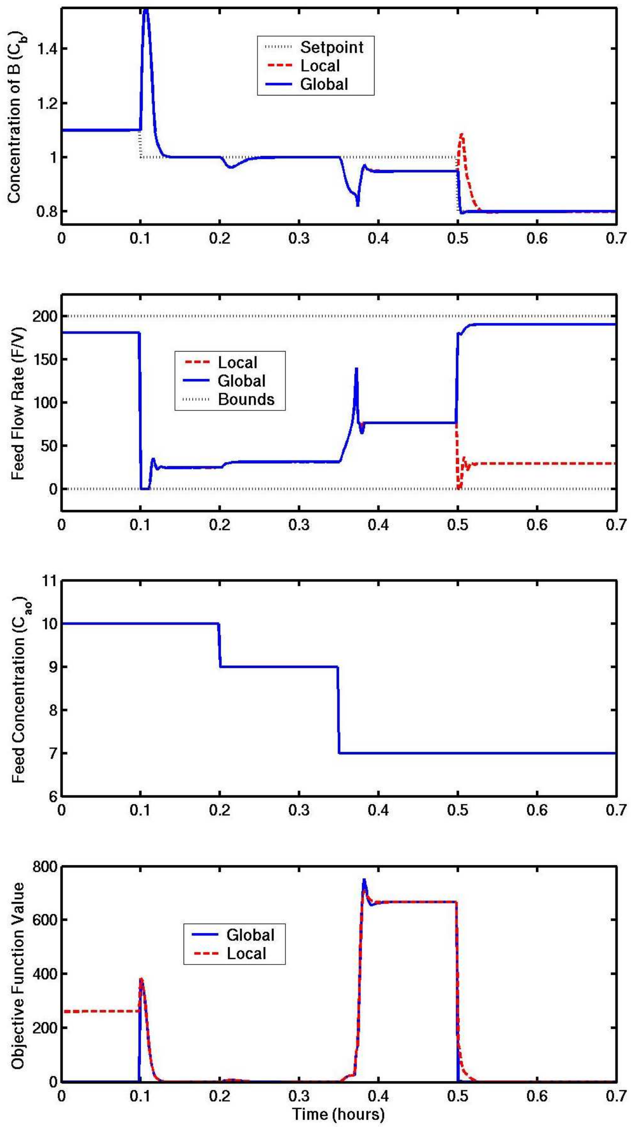
**Tunings:** **m = 2** **$\Gamma_y$=100**

**p = 30** **$\Gamma_u$=0.005**

**e(p)=0**

**Improvement over using the traditional objective function without the hard constraint.**

# Noise Run




## Objective Function

$$\Phi = \sum_{i=1}^{p-1} \Gamma_y e(i) + \Gamma_T e(p) + \sum_{i=1}^{m} \Gamma_u \Delta u(i)$$

**e(p) weighted more heavily than other errors.**

## Tunings

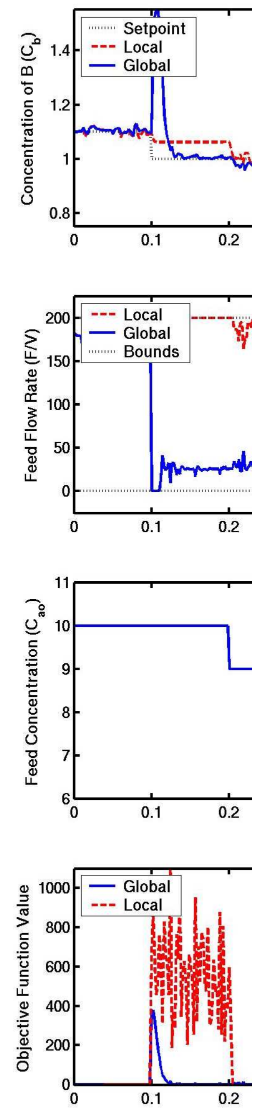
**m = 2** $\Gamma_y = 100$

**p = 30** $\Gamma_u = 0.005$

$\Gamma_T = 10{,}000$

## Noise

- white measurement noise with a standard deviation of 3%



# Summary

- A NMPC algorithm using a deterministic global optimization search method is proposed.
- The deterministic approach guarantees global optimums to the nonconvex NLPs associated with the NMPC formulation.
- The algorithm eliminates poor performance in the CSTR example resulting from suboptimal input trajectories provided by local solution techniques.
- The proper objective function and controller tunings must be utilized to achieve the desired closed-loop results.
- Considerations must be made for cases where the desired solution cannot be obtained sufficiently fast for real-time use.

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