Basic Procedures for Common Problems

ECHE 550, Fall 2002

Steady State Multivariable Modeling and Control

1. Determine what variables are available to manipulate (inputs, Δu) and what variables are available to measure (outputs, Δy)

2. Note how many input and output variables you have.

3. Start to write equations for the **output variables**. This means write something in the form:

$$\begin{aligned} \Delta y_1 &= ??? \\ \Delta y_2 &= ??? \\ \vdots & \quad \vdots \\ \Delta y_n &= ??? \end{aligned}$$

4. Read through the problem and establish relationships between individual inputs (Δu_i) and individual outputs (Δy_j). The relationships generally represent the **gain** of the individual input output relationship, for example $\Delta y_j = K \Delta u_i$. For example: “Changing input 1 by 2% decreases output 1 by 5” means $\Delta u = 2\%$ and $\Delta y = -5$ and

$$-5 = K2$$

Or $K = -5/2$ and $\Delta y_1 = -2.5 \Delta u_1$.

5. Put all of the relationships into the equations. Keep reading through the word expression until you relate all specified inputs and outputs:

$$\begin{aligned} \Delta y_1 &= -2.5\Delta u_1 + ??? \\ \Delta y_2 &= 4??? \\ \vdots & \quad \vdots \\ \Delta y_n &= ??? \end{aligned}$$

6. Write out the equations with all input variable in every equation, even if they have a 0 coefficient.

$$\begin{aligned} \Delta y_1 &= -2.5\Delta u_1 + 0\Delta u_2 + 3\Delta u_3 \\ \Delta y_2 &= 0\Delta u_1 + 4\Delta u_2 + 1\Delta u_3 \\ \Delta y_3 &= 5\Delta u_1 + 10\Delta u_2 + 2\Delta u_3 \end{aligned}$$

7. Realize that this can be put in the form:

$$\underline{\Delta y} = \underline{\underline{K}}\,\underline{\Delta u}$$

Dynamic Modeling

1. Try to figure out what is changing with time. Try to figure out what are manipulated inputs ($u_i(t)$), what are disturbances ($d_i(t)$) and what are measurements ($y_i(t)$).

2. Start to write dynamic mass and energy balances for the items that are changing.

3. Note the accumulation term

   (a) Changing volume: $V(t) = Ah(t) \;\; \rightarrow \;\; A\frac{dh}{dt}(t)$

   (b) Changing amount of species in a tank: $VC_A(t) \;\; \rightarrow \;\; V\frac{dC_A}{dt}(t)$

   (c) Changing temperature in a tank: $V\rho C_p(T(t) - T^*) \;\; \rightarrow \;\; V\rho C_p\frac{dT}{dt}(t)$

4. Don't forget reaction terms for reacting systems. $Vr(t)$ where $r(t)$ is the reaction rate, usually in the form $r(t) = kC_A(t)$ (or more complex).

5. Write your equations and check units.

State Space

1. Identify $\underline{x}$ as the values that are changing with time in your accumulation term.

2. Identify your manipulated inputs $\underline{u}$.

3. Identify your measurement equations. Your measurements should be expressed as functions of the states and inputs.

4. Write your dynamic equations, including terms for every state and input (with 0 coefficient if necessary).

5. Reorder the terms in you dynamic equations such that states come first in order, then inputs. For example:

$$\frac{dx_1}{dt} = 2x_1 + 3x_2 + 0x_3 + 2u_1 + 5u_2$$

6. Put the dynamic equations in the form

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u}$$

7. Write your measurement equations, including terms for every state and input (with 0 coefficient if necessary).

8. Put your measurement equations in the form:

$$\underline{y} = \underline{\underline{C}}\,\underline{x} + \underline{\underline{D}}\,\underline{u}$$

Laplace Transform of Dynamic Equations

1. If your steady state values are not all = 0, take your dynamic model equations and establish the steady state values for you inputs, states, and outputs. This is accomplished by solving for unknowns with the accumulation terms = 0.

2. If your equations are nonlinear, **linearize** your equations.

$$A\frac{dh}{dt}(t) = F_{in}(t) - \sqrt{h(t)}$$

   Here, $\sqrt{h(t)}$ is nonlinear. Near steady state, it can be approximated as

$$\sqrt{h(t)} \simeq \sqrt{h_{ss}} + \frac{1}{2}h_{ss}^{-\frac{1}{2}}\left(h(t) - h_{ss}\right)$$

   such that

$$A\frac{dh}{dt}(t) = F_{in}(t) - \left(\sqrt{h_{ss}} + \frac{1}{2}h_{ss}^{-\frac{1}{2}}\left(h(t) - h_{ss}\right)\right)$$

3. Subtract the steady state model equations from the dynamic model equations to put everything in **deviation variables**. For example, $y(t) = h(t) - h_{ss}$ and $u(t) = F_{in}(t) - F_{inss}$.

   (a) Remember to express the accumulation term with your deviation variables. For $y(t) = h(t) - h_{ss}$, taking the derivative, $\frac{dy}{dt}(t) = \frac{dh}{dt}(t)$ because h_{ss} is constant.

4. Express your dynamic problem using deviation variables $u(t)$, $y(t)$, $d(t)$. These functions of time should = 0 at time $t = 0$.

5. Take the Laplace transform of your system.

6. Solve algebraically to get in the form

$$y(s) = g(s)\,u(s)$$

   or

$$\frac{y(s)}{u(s)} = g(s)$$

7. If you have disturbances and inputs, your model can look like

$$y(s) = g(s)\,u(s) + g_d(s)\,d(s)$$

   Note that to get $g(s)$ you can assume $d(s) = 0$ then solve for $g(s)$. To get $g_d(s)$ you assume $u(s) = 0$ and solve for $g_d(s)$.

8. If you multiple inputs inputs, your model can look like

$$y(s) = g_1(s)\,u_1(s) + g_2(s)\,u_2(s)$$

9. If you have multiple inputs and multiple measurements, your model can look like

$$\begin{aligned} y_1(s) &= g_{11}(s)\,u_1(s) + g_{12}(s)\,u_2(s) \\ y_2(s) &= g_{21}(s)\,u_1(s) + g_{22}(s)\,u_2(s) \end{aligned}$$

10. Given the input as a function of time $u(t)$ (or input and disturbances) you can determine $u(s)$ (or $u(s)$ and $d(s)$).
11. Plug in to get an expression for $y(s)$ in terms of the variable s

Laplace of Complex Functions

1. You should be familiar with basic functions of time (step, impulse, ramp, exponential decay, sinusoid).
2. If the function is not 0 for $t < 0$ you should put the function in deviation variables. For example, a step in $F_{in}(t)$ at time 0 from 2 to 3 can be expressed as a unit step in $u(t)$ at time 0 with $u(t) = F_{in}(t) - F_{inss}$
3. You should be able to express the complex function as a single function of time. Multiply by the Heaviside function if needed. For a function that ramps from 0 with a slope of 2 until time 10 settling out at a value of 20, this can be expressed as

$$f(t) = 2\,t\,H(t) \;+\; (-2)\,(t-10)\,H(t-20)$$

4. Sketch the individual terms in your function as functions of time, then add them together to check your formulation. You can plug in numbers to check your function.
5. For each term, shift it in time such that the "event" occurs at time zero and determine the Laplace transform. Use the time shift operator if necessary to express the function as some $f(s)$. For the example:

$$f(s) = \frac{2}{s^2} + \frac{-2}{s^2}e^{-20s}$$

Solving for $y(t)$

1. Establish $y(s)$ as a function of s. (Develop dynamic model, take Laplace of model, and determine $u(s)$ and $d(s)$ if needed)
2. Your response may be in the form

$$y(s) = \frac{N_1(s)}{D_1(s)} + \frac{N_2(s)}{D_2(s)}e^{-\alpha s} + ... + \frac{N_3(s)}{D_3(s)}e^{-\beta s}$$

   This expression with multiple terms will be treated as multiple different responses, each shifted in time.
3. If you have a time delay, $e^{-\alpha s}$, ignore it for now.
4. Take a term from $y(s)$ and determine the **poles,** the roots of $D_i(s)$.
5. Perform a **Partial Fraction Expansion** on the term. For expressions with unique poles p_i the result looks like:

$$\frac{N_1(s)}{D_1(s)} = \frac{Z_1}{(s-p_1)} + \frac{Z_2}{(s-p_2)} + ... + \frac{Z_n}{(s-p_n)}$$

   For non-unique poles or imaginary roots, check the Appendix. Non-unique Poles will result in

$$\frac{Z_1}{(s-p_1)} + \frac{Z_2 s}{(s-p_1)} + \frac{Z_3 s^2}{(s-p_1)}$$

   while imaginary roots result in sin or cosine in your $y(t)$

6. Now you should be able to determine the inverse Laplace transform of each expression to yield a function of time, $y_1(t)$.

$$y_1(t) = Z_1 e^{-p_1 t} + Z_2 e^{-p_2 t} + ... + Z_n e^{-p_n t}$$

7. If you had a time delay in your term, shift the response by the time delay:

$$y_1(t) = \left(Z_1 e^{-p_1 (t-\alpha)} + Z_2 e^{-p_2 (t-\alpha)} + ... + Z_n e^{-p_n (t-\alpha)} \right) H(t-\alpha)$$

8. Do this procedure for all your terms in the original $y(s)$

9. Add up all $y_i(t)$ to get $y(t)$