

ECHE 550
Fall 2003
Supplemental Course Notes

Edward P. Gatzke

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Chapter 1

Mathematics Review

Objectives

This is a review of various mathematical topics that you probably have seen in previous mathematics courses. Complete the problems where indicated with “**EXERCISES**”. Some topics are just mentioned, without specific review questions.

1.1 Function of One Variable

You should understand the basic concept of a mathematical function / algebraic function. In this course, we examine process dynamics. Things change with time, $f(t)$. In other cases, you could have a parameter that changes with temperature, like a chemical reaction rate. The mathematical function provides a mapping. A *scalar* function maps one value to another value. $f(x) : \mathbb{R}^1 \rightarrow \mathbb{R}^1$. This can be considered a input-output relationship. Some people also use the analogy of a black-box” you put some number in (*independent variable*) , another comes out (*dependent variable*).

Examples as functions of time:

$$f(t) = \sin(5t)$$

$$f(t) = 2t$$

$$f(t) = e^{3t}$$

Examples including constants

$$f(t) = ce^{\frac{t}{\tau}}, \tau > 0, c > 0$$

Reaction rate k as a function of temperature T :

$$k(T) = k_0 e^{\frac{E}{RT}}$$

Note that k_0 , R , and E are constants.

You should know how to graph functions by hand, specifically any function of time (time as independent variable). For some functions, you may want to pick a variety of values of t and evaluate the function values, then graph $f(t)$ vs. t . For a sum of functions $f = f_1 + f_2$ you can plot f_1 and f_2 and add them point by point. When you multiply two functions, $f = f_1 f_2$ you can graph f_1 and f_2 then multiply them at each point.

1. **EXERCISE**, graph the following functions by hand:

- (a) $f(t) = e^{3t}$
- (b) $f(t) = e^{-3t}$
- (c) $f(t) = e^{-0.3t}$
- (d) $f(t) = \sin(t) + 2t$
- (e) $f(t) = 2t + t^2$
- (f) $f(t) = e^t - t^3 + \sin(t) - 1$
- (g) $f(t) = t(\sin(t))$

Function of Two Variables

Sometimes, a value will be a function of multiple different values. Again, the mathematical function provides a mapping. A function can also map one value to another value. $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}^1, n > 0$.

Example, in a topographic (elevation) map, elevation is a function of map position x, y , $ELEVATION = z = f(x, y)$ or $z = -(x^2 + y^2)$.

Example, reaction rate expression as a function of concentrations and temperature:

$$r = f(C_A, C_B, T) = 3.0 e^{\frac{3}{8.14T}} C_A^2 C_B$$

2. Exercise:

- (a) What is the function describing points on a circle of radius r as a function of x and y ?
- (b) What is the function describing points on a sphere of radius r as a function of x, y , and z ?
- (c) Assuming an ideal gas, what is the function for pressure of a gas as a function of volume, temperature, and moles?

Solving Equations of One Variable

If you have a function of one variable, you may be able to find a *solution* to the equation $f(x) = 0$. This means you find a value of x that *satisfies* the equation. The values of x that satisfy the equation are also called the *roots* of the equation.

Sometimes you can easily solve the equation *analytically*. This means that you get a closed-form expression for the solution that satisfies the equation. For the function $f(x) = x^3$, $x = 0$ is the solution to the equation. For the quadratic equation, $ax^2 + bx + c$, the roots are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

There are a variety of numerical methods to find roots. You may have a very complex nonlinear expression, $f(x)$, that is not easily factored into roots or solved directly. To solve $f(x) = 0$ you can graph the expression, then examine the graph to locate *zero crossings* at the values of x that satisfy the function. Using the bisection method, you can evaluate the function at two points, x_L and x_R , $x_R \geq x_L$. Assuming that $f(x_L) \leq 0$ and $f(x_U) \geq 0$, you know a root must lie in the region $x_L \leq x \leq x_R$. Bisect the region to find $x_M = x_L + \frac{x_R - x_L}{2}$ and evaluate the function at x_M . Update bounds, keeping the region that contains a solution.

3. EXERCISE: Find analytically solutions to the following equations:

- (a) $f(x) = (x - 3)(x^2 - x + 12)$
 (b) $f(x) = (x^2 + 6x + 8)(x - 4)x$

4. **EXERCISE:** Find numerical solutions that satisfy the following equations.

- (a) $f(x) = e^x - x^3 + \sin(x) - 1$, multiple different solutions, x in radians
 (b) $f(\omega) = \pi + \tan^{-1}(20\omega) - 2\omega$, ω in radians

Solving Equations of Multiple Variables

In some cases, you have multiple unknown values. Using a degree of freedom analysis, you must have as many independent equations as unknown values in order to find a solution. For example, given the following equations:

$$\begin{aligned} 3 &= x + e^y \\ 2 &= yx \end{aligned}$$

You can say that $x = 1 - e^y$ using equation 1, then $y = \frac{2}{x}$ or $y = \frac{2}{1 - e^y}$. Now, $f(y) = \frac{2}{1 - e^y} - y$ and try to find y such that $f(y) = 0$, if it exists. Alternatively, for a 2D nonlinear case, you can plot $f_1(x, y) = 0$ and $f_2(x, y) = 0$ and determine the points where the two lines intercept.

5. **EXERCISE:** Find solutions to the following equations:

$$\begin{aligned} 3 &= x^2y \\ 4 &= x + \frac{1}{y} \end{aligned}$$

Check your solution to make sure your values for x and y satisfy both equations.

6. **EXERCISE:** Find solutions to the following equations:

$$\begin{aligned} 4 &= x^2 + y^2 \\ 0 &= x^2 - y \end{aligned}$$

Solving Equations of One Variable, Linear Case

7. You must be able to find the slope (derivative) of a function, $\frac{df}{dt}(t)$ or $\frac{df}{dx}(x)$ given the function and know the derivative of simple functions

(a) **EXERCISE, calculate the derivative of the following functions:**

$$f(t) = \sqrt{t}, f(t) = e^t, f(t) = \sin(at)$$

(b) Chain rule

EXERCISE, calculate the derivative of following function: $f(x) = (e^{ax})^2$

8. Derivative evaluated at a point, $\left.\frac{df}{dt}(t)\right|_{t=ts}$ is the slope of the function $f(t)$ at time $t = ts$. Also defines the slope of the line tangent to $f(t)$ at time $t = ts$.

(a) **EXERCISE** Graph $f(t) = t^2 + t$

(b) **EXERCISE** Value of function and slope for $f(t) = t^2 + t$ at $t = 0, t = -1, t = 1$

9. Basic algebra, distributive property, $ax + bx = (a + b)x$

10. Basic algebra, solving for a variable.

Solve for x in equation $ax = by + cx$ with constants a, b, c . First, get terms with x on one side: $ax - cx = by$, use distributive property: $(a - c)x = by$, divide to solve for x in terms of y and some constants: $x = \frac{by}{(a-c)}$

11. Adding two fractions with different denominators.

12. Partial Fraction Expansion of fractions with polynomials in the numerator and denominator.

EXERCISE Find A and B in

$$\frac{5x + 2}{(2x + 1)(3x + 2)} = \frac{A}{2x + 1} + \frac{B}{3x + 2}$$

13. Finding roots of a polynomial. Given $f(x)$, what are values of x to make $f(x) = 0$?

(a) General case, $(x - r_1)(x - r_2) \dots (x - r_n) = 0$, roots $= r_1, r_2, \dots, r_n$.

(b) Second order case, quadratic equation for: $ax^2 + bx + c = 0$ with 2 roots at $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

14. Determinant of a matrix.

(a) **EXERCISE:** What is the determinant of

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

(b) **EXERCISE:** What is the determinant of

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

(c) **EXERCISE:** What is the determinant of

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

15. Multiple linear equations

Solution of multiple linear equations by row reduction / Gaussian elimination

(a) **EXERCISE:** Solve the following set of linear equations by hand:

$$1x + 1y + 1z = 0$$

$$1x + 2y + 3z = 1$$

$$3x + 3y + 1z = 2$$

16. Scalar values. Numbers can be a constant scalar (3, -0.1, e , π) or a variable scalar (x , y , z)

17. Vector values. Examples in 3 dimensions, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Note you are not limited to 3 dimensions:

$$\begin{bmatrix} C_A \\ C_B \\ C_C \\ C_D \end{bmatrix}$$

18. Multivariable functions . Given points in $x \times y$, realize that $f(x, y)$ gives values that can be plotted in 3 dimensions.

EXERCISE: Try to sketch $f(x, y) = x^2 + y^2$ in 3D.

19. Partial Derivative / Gradient of a multivariable function

(a) Partial derivative as slope of tangent surface in direction of one variable

EXERCISE: What is $\frac{\delta f}{\delta x}$ of $f(x, y) = x^2 + y^2$?

(b) **EXERCISE:** What is $\frac{\delta f}{\delta x}$ of $f(x, y) = x^2 + y^2$ evaluated at $x = 1, y = 1$?

(c) **EXERCISE:** What is the gradient of $f(x, y) = x^2 + y^2$?

$$\begin{bmatrix} \frac{\delta f}{\delta x} \\ \frac{\delta f}{\delta y} \end{bmatrix}$$

(d) **EXERCISE:** What is the equation of the plane tangent to $f(x, y) = x^2 + y^2$ at $x = 2, y = 1$?

20. Integration of a function

(a) Integration of basic functions $f(x) = x^2, e^{ax}, \sin(ax), \cos(bx)$

(b) Integration by parts

21. Basic differential equations such as:

$$\frac{dy}{dx} = y$$

This can be solved by separation of variables,

$$\frac{dy}{y} = dx$$

Integrating to get $\ln(y) = x + c$. Assuming $c = 0$, $y = e^x$ where y is a function of x . The same differential equation can be put in the form

$$\frac{df}{dt}(t) = f(t)$$

with the solution $f(t) = ce^t$. Obviously, given that you know $f(t) = ce^t$, $\frac{df}{dt} = ce^t$, so $f(t) = ce^t$ is the differential equation solution, where the constant c can be found from initial conditions for $f(t)$ or $\frac{df}{dt}(t)$

22. **EXERCISE:** Go to <http://www.ncsu.edu/felder-public/ILSdir/ilsweb.html> and take the learning style test. **Record your four results.** Go to <http://www.ncsu.edu/felder-public/ILSdir/styles.htm> and read about your learning style.
23. **EXERCISE: Send Dr. Gatzke an email: gatzke@sc.edu** Please include your permanent home address and phone number for future survey information.

Chapter 2

Matlab

Objectives

- Learn basic functionality of Simulink and Matlab
- Solve systems of linear equations ($Ax = b$) using Matlab
- Simulate a simple dynamic system using Simulink

2.1 Simulink Tutorials

Links to the following Simulink Tutorials can be found at the course web page under the resources link. The course web page is now online at: <http://opus.che.sc.edu/~gatzke/courses/eche550>. Work through these tutorials at your own pace.

<http://www.messiah.edu/acdept/depthome/engineer/Resources/tutorial/matlab/simu.html>

http://www.rpi.edu/dept/chem-eng/WWW/faculty/bequette/lou/simtut/simtut_html.html

2.2 Matlab Command Prompt

Commands can be issued at the Matlab command prompt. For example, the command `ver` shows information about the installed Matlab components, referred to as *toolboxes*.

2.2.1 Data Types and Assignment Statements

Matlab supports scalar, vectors (row and column), matrices, strings, multidimensional arrays, complex structures and many other data types. Try the following examples:

```
> >a=1  
a =  
1
```

```
>> b=[2 3 4]
b =
    2    3    4
>> c=[1 2 ; 3 4]
c =
    1    2
    3    4
>> d='string'
d =
    string
```

Note that a variable name does not have any information associated with it until a value is assigned. Until the command `b=[2 4]` is entered, `b` holds nothing in memory. To access a portion of a matrix, the `:` operator can be used: `b(2:3)` returns the second and third elements of the vector `b`. The command `c(:,1)` returns the rows for column 1 of matrix `c`.

The following are special data structures: multidimensional arrays and structures.

```
>> e(2,2,2)=1
e(:, :, 1) =
    0    0
    0    0
e(:, :, 2) =
    0    0
    0    1
>> f.name='Bob'
f =
    name:    'Bob'
>> f.age=37
f =
    name:    'Bob'
    age:     37
>>
```

Note that the multidimensional array can be used to hold multiple matrices of the the same size. The command `e(:, :, 1)` returns the rows and columns of the first matrix. The command `e(:, 1, 1)` returns the rows of the first column of the first matrix. These values can be set to a single numeric value using a command such as `e(:, 1, 1)=6` or `e(:, 1, 1)=[1;2]`. The structure data type allows you to store different types of information with a single variable name.

The command `whos` will provide information about the values stored in memory

```
>>
>> whos
Name Size Bytes Class
a 1x1      8 double array
b 1x2     16 double array
c 2x2     32 double array
d 1x6     12 char array
e 2x2x2   64 double array
f 1x1    262 struct array
Grand total is 27 elements using 394 bytes
>>
```

Matlab accepts normal math commands:

- `+ - * / ^`

- `sin()` `cos()` `tan()`
- `exp()` `log()` `log10()`
- Matrix commands are supported for `+` `-` `*` (assuming the matrices are conformable). The slash operator can be used for matrices with B/A being roughly $B*INV(A)$.
- Element by element multiplication can be used with `.*`. For example $[1\ 2\ 3]*[1\ 2\ 3]$ is not conformable, but $[1\ 2\ 3].*[1\ 2\ 3]$ would return $[1\ 4\ 9]$
- Element by element power terms can be used with `.^`. For example $[1\ 2\ 3].^3$ is would return $[1\ 16\ 27]$.
- The `^` operator can be used for square matrices, $A^3=A*A*A$

Some functions produce a single output, such as `sine(3)`. Others can produce multiple values for output, depending on the calling sequence. The following examples use the `size` command for a 1x2 matrix (row vector). Calling `size` with one command produces an array with the row and column sizes, calling with two values stores the scalar values in the arguments. Use help to find information on functions

```
>> size(b)
ans =
1 2
>> [rows,cols]=size(b)
rows =
1
cols =
2
>>
```

Other useful commands include

- `size` to determine the size of the dimensions in a vector or matrix
- `help` to provide help for a specific command
- `pwd` to print the current directory
- `cd` to change directories
- `ls` to list the contents of a directory
- `which` to determine if a variable name is already used as a variable or a function
- `clear` to clear a variable (or the entire workspace)
- `UpArrow` to use command recall, or type the first letter the use `UpArrow`
- `path` to specify what the execution path is
- `;` semicolon suppresses output to the screen

- `%` is used to comment out a line
- `logspace` can be used to produce a vector
- `poly` can be used to calculate polynomial coefficients given roots of the polynomial expression. Note that Matlab represents polynomials as a vector of coefficients, so $x^2 + 3x + 2$ would be represented as `[1 3 2]`.
- `conv` can be used to multiply (convolve) two polynomials
- `roots` can be used to find roots of a polynomial

Some shortcuts are available for assigning vectors:

```
>> a=[1:.1:1.5]
a = 1.0000 1.1000 1.2000 1.3000 1.4000 1.5000
```

2.2.2 Flow Control Statements

Matlab can use flow control statements such as IF, WHILE, and FOR. Use *help for* or *help while* or *help if* for more information.

```
>> flag=0;
>>i=0;
% WHILE LOOP EXAMPLE
% WITH IF STATEMENT
while flag==0
    i=i+1
    if i>2
        flag=1;
    end
end
i =
    1
i =
    2
i =
    3
% FOR LOOP EXAMPLE
for i=1:3
    a(1,i)=i*i
end
a =
    1
a =
    1 4
a =
    1 4 9
```

2.2.3 Plotting

Matlab is a very powerful visualization tool!

To plot 20 random points at space by 5 units on the x axis:

```
x=[1:5:100]
```

```
y=rand(20,1)
```

```
plot(x,y)
```

```
plot(x,y,':')
```

```
plot(x,y,'o',x,2*y,'o')
```

To make a 3D surface plot:

```
mesh(rand(5,5))
```

Other useful commands

- title
- ylabel
- xlabel
- gtext
- legend
- plot3
- loglog
- semilogx, semilogy
- subplot
- axis

2.2.4 Matlab M-files

Groups of commands can be put into a script file, usually with a .m filename extension. The Matlab command *edit* should start a text editor for editing these files. Put the following commands in a file named *test.m* then call test at the Matlab prompt to run the commands.

```
k=7;  
tau=8;  
w=logspace(-4,4,100);  
subplot(2,1,1)  
g=(k./(tau*j*w+1))  
loglog(w,abs(g))  
ylabel('Amplitude Ratio')  
subplot(2,1,2)  
semilogx(w,angle(g)*180/pi)  
xlabel('Frequency')  
ylabel('Phase Angle, degrees')
```

2.2.5 Matlab Function Files

Just as Matlab has defined functions and procedures, a user can specify a user function. See *help* function for more information. The function must be contained in a file named *function.m* (where function is the name of the function).

Chapter 3

Linear Algebra

Objective

Demonstrate solution methods for systems of linear equations. Show that a system of equations can be represented in matrix-vector form.

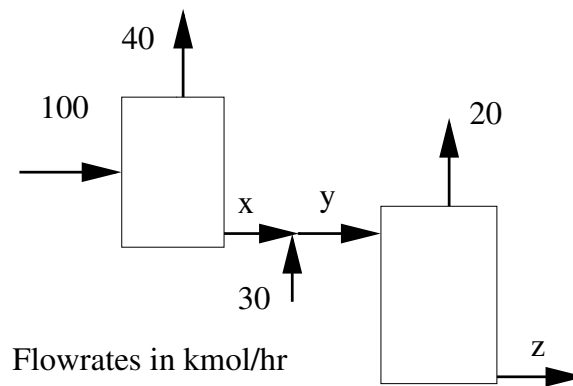


Figure 3.1: Two distillation columns in series.

3.1 Example System

Two distillation columns in series with a additional feed stream mixing in with the bottoms stream of the first column. The flowrate of three streams are unknown. As indicated in the Figure, the flowrate of stream x , y , and z are unknown. No reaction is taking place. The steadystate flowrates must be calculated.

Basic Mass Balance:

$$accumulation = in - out + created - destroyed$$

Mass Balance on first column (In this case, assume steady state: $accumulation = 0$):

$$0 = 100 - 40 - x$$

Mass balance on mixing point:

$$0 = x + 30 - y$$

Mass balance on second column:

$$0 = y - 20 - z$$

Three linear equations:

$$0 = 100 - 40 - x$$

$$0 = x + 30 - y$$

$$0 = y - 20 - z$$

Note these are linear equations. The unknown variables have constant linear coefficients, nonlinear terms do not appear (no x^2 , no $\sqrt{x}$, no e^x). You can rearrange this set of equations to get all the variable terms on the left side and the constants on the right. After some This set of equations can be written as:

$$\begin{aligned} 1x + 0y + 0z &= 60 \\ -1x + 1y + 0z &= 30 \\ 0x - 1y + 1z &= -20 \end{aligned} \tag{3.1}$$

As we will see later, this can be more compactly written as:

$$\underline{\underline{A}}\underline{\underline{x}} = \underline{\underline{b}}$$

You may already realize that the solution to this problem is $x = 60$, $y = 90$, and $z = 70$. For more complex systems, this is not quite so easy. To solve the three linear equations simultaneously in a general manner, you can perform row reduction using three possible row operations:

RULES

1. Add (or subtract) one row to (or from) another
2. Multiply or divide a row by a scalar value (any real scalar $\neq 0$)
3. Swap position of rows

Typically you would perform these operations until you have a triangular representation (all 0's above or below the diagonal). The triangular form allows for quick solution.

The set of linear equations in Equation 3.6 can be compactly written using only the coefficients as:

$$\begin{array}{ccc|c} 1 & 0 & 0 & 60 \\ -1 & 1 & 0 & 30 \\ 0 & -1 & 1 & -20 \end{array}$$

We need to perform steps 1-3 to get the system of equations in triangular form with ones on the diagonal and zeros below the diagonal, like

$$\begin{array}{ccc|c} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{array}$$

We can look at the original system of equations and realize that we must get zeros in position 2,1 (row 2, column 1) and position 3,2 (row 3, column 2). You can multiply row 2 by -1 using Rule 2:

$$\begin{array}{ccc|c} 1 & 0 & 0 & 60 \\ 1 & -1 & 0 & -30 \\ 0 & -1 & 1 & -20 \end{array}$$

Next, swap position of rows 2 and 3 using Rule 3 to get:

$$\begin{array}{ccc|c} 1 & 0 & 0 & 60 \\ 0 & -1 & 1 & -20 \\ 1 & -1 & 0 & -30 \end{array}$$

Then, subtract row 1 from row 3 using Rule 1 to get:

$$\begin{array}{ccc|c} 1 & 0 & 0 & 60 \\ 0 & -1 & 1 & -20 \\ 0 & -1 & 0 & -90 \end{array}$$

Then, multiply rows 2 and 3 by -1 using Rule 2:

$$\begin{array}{ccc|c} 1 & 0 & 0 & 60 \\ 0 & 1 & -1 & 20 \\ 0 & 1 & 0 & 90 \end{array}$$

Subtract row 2 from row 3 using Rule 1 again to get:

$$\begin{array}{ccc|c} 1 & 0 & 0 & 60 \\ 0 & 1 & -1 & 20 \\ 0 & 0 & 1 & 70 \end{array}$$

Now, all coefficients below the diagonal are 0. The solution can be found quickly. From equation 3 (row 3), $z = 70$. Using equation 2 (row 2) $y - z = 20$, but you know that $z = 70$ so $y = 90$. Equation 1 (row 1) gives $x = 60$, so the overall solution is $x = 60$, $y = 90$, and $z = 70$.

CHECK SOLUTIONS: You can plug your solution back into the original three equations and verify that the equations are satisfied. **THIS WILL HELP YOU ON EXAMS.**

Note that the general Gaussian elimination or row reduction method specifies that you start with column 1 and perform operations until all coefficients below the diagonal are 0, then move to column 2 and perform operations until all coefficients below the diagonal are zero, etc.

3.2 Linear Equations - Special Cases

In general, there are three possibilities for a “square” set of linear equations.

3.2.1 Case A - One solution

Consider a simpler system: $x + y = 1$ and $x - y = 1$. Graphically, you can plot the two lines and look for the intersection of two lines which occurs at $x = 1, y = 0$. The system of equations is:

$$\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 1 \end{array}$$

Subtracting row 1 from row 2 gives:

$$\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 0 \end{array}$$

This implies $-2y = 0$ or $y = 0$ and $x + y = 1$ or $x = 1$ as you already realized.

In 3 dimensions (3 unknowns) each row represents a plane. Two equations can intersect to give a line, and a line can intersect with a third plane to give a point, the single solution (in a single solution case).

3.2.2 Case B - No solution

Consider the system $x + y = 1$ and $x + y = 2$. Graphically, this represents two lines that never intersect.

$$\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 2 \end{array}$$

Note that column 1 and column 2 are identical. Subtracting row 1 from row 2 gives:

$$\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 1 \end{array}$$

You know that $0x + 0y = 1$ cannot be true. For a “square” system, if Gaussian elimination results in a 0 on the diagonal, this may be the case.

3.2.3 Case C - Many solutions

Consider the system $x + y = 1$ and $2x + 2y = 2$. Graphically, this represents two lines that are coincident.

$$\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}$$

Subtracting twice the value of row 1 from row 2 gives:

$$\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}$$

These equations are consistent. $0x + 0y = 0$ and $x + y = 1$ are consistent. There is no single solution, as many solutions make the equation $x + y + 1$ consistent.

1. transfer function relating y_{sp} and y .

3.3 Nonsquare Systems

The original example was for a “square” system with 3 unknowns and 3 equations. You may often end up with more (or fewer) equations than unknowns.

Consider the original set of equations:

$$\begin{aligned}1x + 0y + 0z &= 60 \\ -1x + 1y + 0z &= 30 \\ 0x - 1y + 1z &= -20\end{aligned}$$

One additional equation can be specified using a mass balance on the entire system, $0 = 100 + 30 - 40 - 20 - z$.

$$\begin{aligned}1x + 0y + 0z &= 60 \\ -1x + 1y + 0z &= 30 \\ 0x - 1y + 1z &= -20 \\ 0x + 0y + 1z &= 70\end{aligned}\tag{3.2}$$

These four linear equations are not “linearly independent.” You can test this by using row operations to make two rows identical. Simultaneously adding row 1 and row 3 to row 2 will make row 2 the same as row 4.

$$\begin{aligned}1x + 0y + 0z &= 60 \\ 0x + 0y + 1z &= 70 \\ 0x - 1y + 1z &= -20 \\ 0x + 0y + 1z &= 70\end{aligned}\tag{3.3}$$

This set of equations can still be satisfied using the original solution $x = 60$, $y = 90$, and $z = 70$.

3.3.1 Reconciliation and Nonsquare Systems

For curve fitting, parameters that appear linearly can be formulated as a nonsquare solution to a linear algebraic system of equations. Given that you have some (scalar valued) measured value, y , that depends on a process parameter, x . Assume the model takes the form:

$$y = mx + b\tag{3.4}$$

Technically, you only need two data points to find m and b , the model parameters. Assuming that you have more than two data points, we often desire to determine the “best-fit” for the line. These parameters

minimize the sum of the square of the model error. For an experiment with four data points:

$$\begin{aligned}y(1) &= m x(1) + b \\y(2) &= m x(2) + b \\y(3) &= m x(3) + b \\y(4) &= m x(4) + b\end{aligned}\tag{3.5}$$

Here, you know values of y and x but m and b are your unknown values. This can be written as a set of equations:

$$\begin{bmatrix} y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix} = \begin{bmatrix} x(1) & 1 \\ x(2) & 1 \\ x(3) & 1 \\ x(4) & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix}$$

3.4 Vectors

A group of unknown (or known) values can be “stacked” to form a vector. In the example problem, the unknowns x , y , and z can be described by the vector $\underline{x}$:

$$\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The solution to the problem has a known value and can be written as a vector $\underline{x}_{soln}$:

$$\underline{x}_{soln} = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

Note that the underbar is used to distinguish between $\underline{x}$ (the vector) and x the unknown. A vector is NOT limited to 2 or 3 unknowns (dimension of the vector).

3.5 The Matrix

A matrix is similar to a vector, having 2 dimensions. One may think of it as a group of vectors augmented together. A Matrix has a size, $m \times n$ representing m rows and n columns. The values for m and n are sometimes written as subscripts for the matrix. For example, the 2x3 matrix $\underline{\underline{A}}_{2 \times 3}$ with two rows and three columns may have values:

$$\underline{\underline{A}}_{2 \times 3} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$$

Note that each of the six elements has two indices. The first index is the row, the second is the column. For the applications in this class, a matrix will have constant coefficient values. Some example matrices:

$$\underline{\underline{A}}_{2 \times 3} = \begin{bmatrix} 0 & -2 & 1 \\ 5 & 1 & 0.2 \end{bmatrix} \quad \underline{\underline{B}}_{3 \times 3} = \begin{bmatrix} 6 & 0 & 0 \\ -2 & 0 & -1 \\ 3 & -1 & 5 \end{bmatrix}$$

Square Matrix - A matrix with indices equal ($m = n$).

Note: A vector can be seen as a special matrix having only 1 column.

Transpose - The transpose operator swaps the indices of a matrix (or vector). For example, for $\underline{\underline{A}}_{2 \times 3}$ as before:

$$\left(\underline{\underline{A}}_{2 \times 3}\right)^T = \begin{bmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \\ a_{1,3} & a_{2,3} \end{bmatrix}$$

Example. For the matrix $\underline{\underline{A}}$

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\underline{\underline{A}}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Finally, one can take the transpose of a vector. For $\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\underline{x}^T = [x \ y \ z] = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T$$

Row Vector - The transpose of a vector is also known as a row vector.

Dot Product - The dot product of two vectors is the sum of the product of the elements taken individually. Examples:

$$\underline{x} \cdot \underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x^2 + y^2 + z^2$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1x + 2y + 3z$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1 \times 4 + 2 \times 5 + 3 \times 6 = 32$$

Matrix Multiplication - Two matrices can be multiplied together. For example $\underline{\underline{A}}_{m \times n}$ can be multiplied by $\underline{\underline{B}}_{n \times j}$. Matrix $\underline{\underline{A}}$ has m rows and n columns, while $\underline{\underline{B}}$ has n rows and j columns.

$$\underline{\underline{A}}_{m \times n} = \begin{bmatrix} \dots & r_1 & \dots \\ \dots & r_2 & \dots \\ & \vdots & \\ \dots & r_m & \dots \end{bmatrix}$$

Here, each row up to r_m is a row vector with n elements.

$$\underline{\underline{B}}_{n \times j} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ c_1 & c_2 & \dots & c_j \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

Here, each column up to column c_j is a vector (column vector) with n elements. To compute $\underline{\underline{A}}_{m \times n} \underline{\underline{B}}_{n \times j}$ or simply $\underline{\underline{A}} \times \underline{\underline{B}}$ or just $\underline{\underline{A}} \underline{\underline{B}}$

$$\underline{\underline{A}}_{m \times n} \underline{\underline{B}}_{n \times j} = \begin{bmatrix} r_1^T \cdot c_1 & r_1^T \cdot c_2 & \dots & r_1^T \cdot c_j \\ r_2^T \cdot c_1 & r_2^T \cdot c_2 & \dots & r_2^T \cdot c_j \\ \vdots & \vdots & & \vdots \\ r_m^T \cdot c_1 & r_m^T \cdot c_2 & \dots & r_m^T \cdot c_j \end{bmatrix}$$

Method - To compute $\underline{\underline{A}}_{m \times n} \underline{\underline{B}}_{n \times j}$, the result will have j columns. The first column of the result is computed by taking the dot product of $\underline{\underline{B}}_{1 \times j}$ (first column of $\underline{\underline{B}}$) with the transpose of all the rows of $\underline{\underline{A}}$. The second column of the result is computed by taking the dot product of $\underline{\underline{B}}_{2 \times j}$ (second column of $\underline{\underline{B}}$) with the transpose of all the rows of $\underline{\underline{A}}$. Repeat up to the j^{th} column of $\underline{\underline{B}}$ which produces the j^{th} column of the result.

Note: In general, $\underline{\underline{A}} \underline{\underline{B}} \neq \underline{\underline{B}} \underline{\underline{A}}$.

Conformable - In order to multiply $\underline{\underline{A}}_{m \times n} \underline{\underline{B}}_{n \times j}$ the “inner” dimensions must be equal. In $\underline{\underline{A}}_{m \times n} \underline{\underline{B}}_{n \times j}$, if the first matrix has n columns and the second matrix must n rows.

Matrix Multiplication Examples:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 5 + 14 & 6 + 16 \\ 15 + 28 & 18 + 32 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -4 + 10 \\ 4 + 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x + 2y \\ x + y \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 4 - 6 & 3 \\ 2 + 2 & -1 \\ 10 + 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 4 & -1 \\ 10 & 0 \end{bmatrix}$$

3.6 Column Example

Consider again the equations from the original distillation column example:

$$\begin{aligned}1x + 0y + 0z &= 60 \\ -1x + 1y + 0z &= 30 \\ 0x - 1y + 1z &= -20\end{aligned}$$

Notice that the variables (with constant coefficients) are on the left side and constant values are on the right hand side. This set of linear equations can be represented in the compact notation $\underline{\underline{A}}\underline{x} = \underline{b}$ where

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} 60 \\ 30 \\ -20 \end{bmatrix}$$

Identity Matrix - The identity matrix has values of one on the diagonal and zeros elsewhere. It is defined as $\underline{\underline{I}}$ and for a square matrix $\underline{\underline{A}} \underline{\underline{I}} = \underline{\underline{A}}$ and $\underline{\underline{I}} \underline{\underline{A}} = \underline{\underline{A}}$.

$$\underline{\underline{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3.6.1 How to solve sets of linear equations

We need a solution to the matrix equation $\underline{\underline{A}}\underline{x} = \underline{b}$. You cannot “divide” by a matrix:

$$\underline{x} \neq \underline{b}/\underline{\underline{A}}$$

There is no “division” operator for a matrix. Instead, an inverse is defined for some square matrices such that

$$\underline{\underline{A}} (\underline{\underline{A}})^{-1} = \underline{\underline{I}}$$

Also,

$$(\underline{\underline{A}})^{-1} \underline{\underline{A}} = \underline{\underline{I}}$$

Now, to solve $\underline{\underline{A}}\underline{x} = \underline{b}$ for $\underline{x}$

First, multiply on the left by $(\underline{\underline{A}})^{-1}$

$$(\underline{\underline{A}})^{-1} \underline{\underline{A}} \underline{x} = (\underline{\underline{A}})^{-1} \underline{b}$$

Realizing that $(\underline{\underline{A}})^{-1} \underline{\underline{A}} = \underline{\underline{I}}$ replace $(\underline{\underline{A}})^{-1} \underline{\underline{A}}$ with $\underline{\underline{I}}$.

$$\underline{\underline{I}} \underline{x} = (\underline{\underline{A}})^{-1} \underline{b}$$

Now, realizing $\underline{\underline{I}} \underline{x}$ is $\underline{x}$, the solution is

$$\underline{x} = (\underline{\underline{A}})^{-1} \underline{b}$$

Note that multiplying on the right will not lead to a solution.

$$\underline{\underline{A}} \underline{x} (\underline{\underline{A}})^{-1} = \underline{b} (\underline{\underline{A}})^{-1}$$

3.6.2 How determine a matrix inverse

To solve $\underline{\underline{A}} \underline{x} = \underline{b}$, you need to know $(\underline{\underline{A}})^{-1}$. We are going to use row reduction to calculate $(\underline{\underline{A}})^{-1}$. Start with $\underline{\underline{A}} \mid \underline{\underline{I}}$. use row reduction techniques until $\underline{\underline{A}}$ is $\underline{\underline{I}}$. $(\underline{\underline{A}})^{-1}$ if it exists will be on the right where $\underline{\underline{I}}$ was originally.

Inverse Example

Solve the following for $\underline{x}$ using $(\underline{\underline{A}})^{-1}$:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \underline{x} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

For this procedure, one must first calculate $(\underline{\underline{A}})^{-1}$. Set up $\underline{\underline{A}} \mid \underline{\underline{I}}$ as:

$$\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}$$

Use row reduction to get

$$\begin{array}{cc|cc} 1 & 0 & ? & ? \\ 0 & 1 & ? & ? \end{array}$$

Then verify that $\underline{\underline{A}} (\underline{\underline{A}})^{-1} = \underline{\underline{I}}$. Use $(\underline{\underline{A}})^{-1}$ to calculate $\underline{x}$ using $\underline{x} = (\underline{\underline{A}})^{-1} \underline{b}$. Verify solution again to be safe.

START

Start by using row reduction on

$$\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}$$

Multiply row 2 by 1/3 to get :

$$\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & \frac{4}{3} & 0 & \frac{1}{3} \end{array}$$

Then subtract row 1 from row 2 to get:

$$\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -\frac{2}{3} & -1 & \frac{1}{3} \end{array}$$

Now, multiply row 2 by -3/2 to get:

$$\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}$$

To get the left side looking like the identity matrix, subtract 2 times row 2 from row 1. Note that this is a compound use of row reduction rules.

$$\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}$$

You now have $(\underline{\underline{A}})^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

Now verify that $\underline{\underline{A}} (\underline{\underline{A}})^{-1} = \underline{\underline{I}}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1(-2) + 2(\frac{3}{2}) & 1(1) + 2(-\frac{1}{2}) \\ 3(-2) + 4(\frac{3}{2}) & 3(1) + 4(-\frac{1}{2}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

You may also verify that $(\underline{\underline{A}})^{-1} \underline{\underline{A}} = \underline{\underline{I}}$

$$\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -2 + 3 & -4 + 4 \\ \frac{3}{2} - \frac{3}{2} & 3 - 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, compute the solution, $\underline{x} = (\underline{\underline{A}})^{-1} \underline{b}$.

$$\underline{x} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -10 + 6 \\ \frac{15}{2} - 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4\frac{1}{2} \end{bmatrix}$$

Again, verify the solution is the solution to the original equations:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \underline{x} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -4 \\ 4\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -4 + 9 \\ -12 + 18 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Just as expected...

3.6.3 Steady State Control Example

Two pumps are used to fill two tanks. The pumps usually operate at 50%, keeping the tanks at levels of 75 inches and 80 inches respectively. It is known that a 1% increase in pump 1 increases the height of tank 1 by 5 inches and the height of tank 2 by 3 inches. For a 1% change in pump 2, the height of tank 2 increases by 4 inches. It is desired to change the operating levels of the tanks to 110 inches and 89 inches.

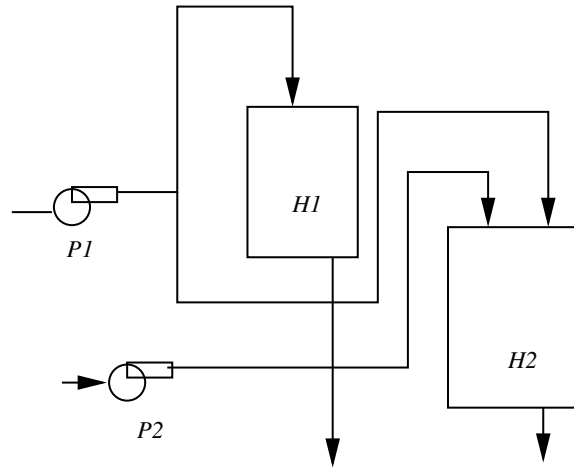


Figure 3.2: Pump / Tank example

What do you know:

$$\begin{aligned} 5 \Delta P_1(\%) &= \Delta H_1(\text{inches}) \\ 3 \Delta P_1(\%) + 4 \Delta P_2(\%) &= \Delta H_2(\text{inches}) \end{aligned}$$

You know the target (reference, setpoint) for H_1 and H_2 as 110 and 89. This translates into $\Delta H_1 = 110 - 75 = 35$ and $\Delta H_2 = 89 - 80 = 9$. You need to increase tank 1 by 35 inches and increase tank 2 by 9 inches. You do not know the final values of the pump speeds. You do know the original steadystate values, 50% and 50%, realizing that:

$$P_{final} = P_{ss} + \Delta P$$

You can now set up linear equations to solve for ΔP_1 and ΔP_2 , then calculate the final values for the pump speeds.

$$\begin{bmatrix} 5 & 0 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} \Delta P_1 \\ \Delta P_2 \end{bmatrix} = \begin{bmatrix} \Delta H_1 \\ \Delta H_2 \end{bmatrix}$$

3.7 Visualization

Each row in $\underline{A}\underline{x} = \underline{b}$ is a single linear equation. For a 2D problem ($\underline{x}$ with 2 elements / unknowns) the equation defines a line in the (x, y) plane. Two equations define two lines, and the unique solution to

$\underline{\underline{A}}\underline{x} = \underline{b}$ is the point $\underline{x}$ where the lines intersect. In some cases, there may be many solutions to $\underline{\underline{A}}\underline{x} = \underline{b}$ and in some cases there may be no solutions to $\underline{\underline{A}}\underline{x} = \underline{b}$.

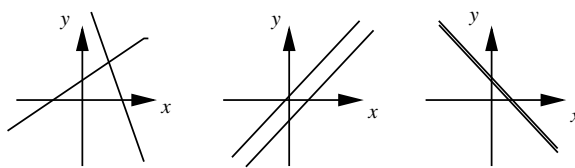


Figure 3.3: Three 2D examples with two equations. Each equation (row) represents a line. The first case has one solution, the second case has no solution, and the third case has many solutions.

For a 3D problem, each row defines the equation for a plane in 3 space. The intersection of 2 non-parallel planes is a line in 3 space, and the intersection of a line and a plane in 3 space is a point. Again, in some cases there may be a single solution, many solutions, or no solutions.

For higher dimensions, each equation defines a **hyperplane** in a n dimensional space, $\mathbb{R}^n$.

3.7.1 Linear Transform

A vector in $\mathbb{R}^n$ means x has n elements. Matrix multiplication of a matrix of size $m \times n$ times a vector of size $n \times 1$ “maps” the vector from $\mathbb{R}^n$ to $\mathbb{R}^m$.

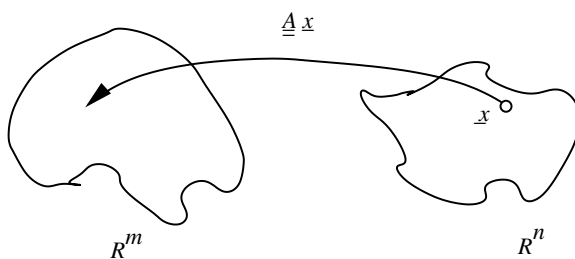


Figure 3.4: Matrix multiplication as a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$.

3.7.2 Range

The range of a matrix is the space of all possible points that may be mapped to in a matrix multiplication of that matrix times an unknown vector.

Range Example 1

For example, the matrix

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

can only map to points on the line $x + y$ in 3D as follows.

$$\underline{\underline{A}}\underline{x} = 2x + 2y + 0z$$

The columns of the matrix define possible directions for the matrix to transform a vector. In this example, columns 1 and 2 are the same, and column 3 is the zero vector. $\underline{\underline{A}}\underline{x}$ where $\underline{x}$ takes any real value will always

be on the line defined by the direction $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Range Example 2

In another example, the matrix

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

can only map to a variety of points in 3D as follows.

$$\underline{\underline{A}}\underline{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} z$$

Again, the columns of the matrix define possible directions for the matrix to transform a vector. In this example, only points in the directions of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ can be reached when multiplying $\underline{\underline{A}}\underline{x}$. These two directions form a plane in 3 dimensional space.

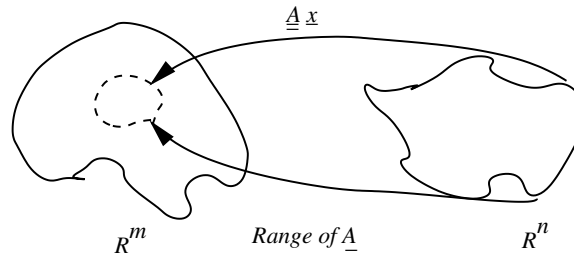


Figure 3.5: Range of $\underline{\underline{A}}$ as space in $\mathbb{R}^m$ of all possible mappings from $\mathbb{R}^n$ using matrix multiplication.

Range Example 3

In another example, the matrix

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

can only map to a variety of points in 3D as follows.

$$\underline{\underline{A}}\underline{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} z$$

Here, column 3 is linearly dependent upon columns 1 and 2. This means that you can find some combination of columns 1 and 2 that give column 3. Column 3 lies in the plane defined by columns 1 and column 2.

Underlying point: For $\underline{\underline{A}}\underline{x} = \underline{b}$ to have a solution, the $\underline{b}$ must be in the range of $\underline{\underline{A}}$.

For the last examples, if $\underline{b} = \begin{bmatrix} ? \\ ? \\ 1 \end{bmatrix}$ (if $\underline{b}$ has element in the z position) there will not be a solution to $\underline{\underline{A}}\underline{x} = \underline{b}$. In such a case, the possible range of $\underline{\underline{A}}$ does not include $\underline{b}$.

Range Example 4

In another example, the matrix

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

can map to all of the points in 3D as follows.

$$\underline{\underline{A}}\underline{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} z$$

Here, column 3 is NOT linearly dependent upon columns 1 and 2. This means that you can find some combination of columns 1, 2, and 3 that give any point in 3 dimensions.

Rank - The rank of a matrix is the number of linearly independent columns. For a square matrix of size $n \times n$, there is a unique solution if there are n independent columns. The matrix would have rank n .

Chapter 4

Laplace Transforms / Deviation Variables

4.1 Simple System Example

Consider a tank draining from an initial height of h_o at time $t = 0$. With no flow into the tank ($F_{in} = 0$) and $F_{out} = \alpha h(t)$ the mass balance can be written:

$$A \frac{dh}{dt}(t) = 0 - \alpha h(t)$$

Moving $\alpha h(t)$ to the left half side and dividing by α gives:

$$\frac{A}{\alpha} \frac{dh}{dt}(t) + h(t) = 0$$

A is the tank area (constant) and α is the proportionality constant for flow out of the tank. These parameters can be replaced by $\tau = A/\alpha$ to give the following differential equation:

$$\tau \frac{dy}{dt}(t) + y(t) = 0 \quad (4.1)$$

The initial tank height at time $t = 0$ can be assumed to be $y(t)|_{t=0} = y_o$. Take the Laplace transform of Equation 4.1:

$$L \left\{ \tau \frac{dy}{dt}(t) \right\} + L \{y(t)\} = 0$$

$L\{y(t)\}$ is easy, $L\{y(t)\} = y(s)$ so we have:

$$L \left\{ \tau \frac{dy}{dt}(t) \right\} + y(s) = 0$$

$L \left\{ \tau \frac{dy}{dt}(t) \right\}$ is a bit more complex. First, you can realize that τ is constant. Convince yourself of this! The L operator on a constant times a function is the same as a constant times the Laplace of the function:

$$L \{c f(t)\} = \int_0^\infty c e^{-st} f(t) dt = c \int_0^\infty e^{-st} f(t) dt = c L \{f(t)\}$$

So you can take the constant value outside the L operator:

$$\tau L \left\{ \frac{dy}{dt}(t) \right\} + y(s) = 0$$

Now, you must remember that $L \left\{ \frac{df}{dt}(t) \right\}$ is just $s f(s) - f(t)|_{t=0}$.

$$\tau (s y(s) - y(t)|_{t=0}) + y(s) = 0$$

And we have initial conditions for the height of the tank, $y(t)|_{t=0} = y_o$

$$\tau (s y(s) - y_o) + y(s) = 0$$

Solving for $y(s)$:

$$\tau s y(s) - \tau y_o + y(s) = 0$$

$$\tau s y(s) + y(s) = \tau y_o$$

$$(\tau s + 1) y(s) = \tau y_o$$

$$y(s) = \frac{\tau y_o}{(\tau s + 1)}$$

Now rearrange a little bit

$$y(s) = \tau y_o \frac{1}{(\tau s + 1)}$$

$$y(s) = \tau y_o \frac{1}{(\tau s + 1)} \frac{1/\tau}{1/\tau}$$

$$y(s) = y_o \frac{1}{(s + \frac{1}{\tau})}$$

This you realize is a constant y_o times the term $\frac{1}{s + \frac{1}{\tau}}$. To get $y(t)$ you must use the inverse Laplace transform, L^{-1} .

$$L^{-1} \{y(s)\} = L^{-1} \left\{ y_o \frac{1}{(s + \frac{1}{\tau})} \right\}$$

Again, y_o is a constant and can be factored out

$$L^{-1} \{y(s)\} = y_o L^{-1} \left\{ \frac{1}{(s + \frac{1}{\tau})} \right\}$$

And we know from lecture that $L \{e^{-at}\} = \frac{1}{s+a}$, so in our case, $a = \frac{1}{\tau}$.

$$y(t) = y_o e^{-(\frac{1}{\tau})t}$$

This is the solution to the original differential equation! Now check your result. At time $t = 0$ your solution for $y(t)$ is $y_o e^{-(\frac{1}{\tau})0} = y_o 1 = y_o$. This matches the initial conditions. The derivative of your result can also be found

$$\frac{dy}{dt}(t) = \frac{d}{dt} \left\{ y_o e^{-(\frac{1}{\tau})t} \right\} = y_o - \left(\frac{1}{\tau} \right) e^{-(\frac{1}{\tau})t}$$

$$\frac{dy}{dt}(t) = -\frac{y_o}{\tau} e^{-(\frac{1}{\tau})t}$$

Plug that back in the original differential EQ, along with your solution for $y(t)$:

$$\tau \frac{dy}{dt}(t) + y(t) = 0$$

$$\tau \left(-\frac{y_o}{\tau}\right) e^{-(\frac{1}{\tau})t} + y_o e^{-(\frac{1}{\tau})t} = 0$$

And we know we have the solution!

4.2 First-Order System Modeling

The first order system model is:

$$\tau \frac{dy}{dt}(t) + y(t) = K u(t)$$

Taking the Laplace transform:

$$\tau s y(s) - \tau y(t)|_{t=0} + y(s) = K u(s)$$

If we assume that $y(t)|_{t=0} = 0$ this simplifies the equation to

$$\tau s y(s) + y(s) = K u(s)$$

We can then solve for $y(s)$

$$(\tau s + 1)y(s) = K u(s)$$

$$y(s) = \frac{K}{(\tau s + 1)} u(s)$$

Here, $\frac{K}{(\tau s + 1)}$ is the process model relating $u(s)$ and $y(s)$. This is sometimes called $g(s) = \frac{K}{(\tau s + 1)}$. Given $u(t)$ you can find $u(s)$, and given a model of your system you can find $g(s)$. Realizing that $y(s) = g(s)u(s)$ you can then find $y(t)$.

From a **process reaction curve** (the data for $y(t)$ and $u(t)$ given a step in the input $u(t)$) you can find the PROCESS GAIN K from the equation:

$$K = \frac{y_{fin} - y_{init}}{u_{fin} - u_{init}} = \frac{\Delta y}{\Delta u}$$

The time constant is a bit trickier. First, lets assume $u(t)$ is a step at time $t = 0$ from a value of 0 to a new value of A . The Laplace transform of the step function is:

$$u(s) = \frac{A}{s}$$

Now, we have enough information to get $y(s)$ and $y(t)$

$$y(s) = \frac{K}{(\tau s + 1)} u(s)$$

$$y(s) = \frac{K}{(\tau s + 1)} \frac{A}{s}$$

To solve this easily, we need the partial fraction expansion:

$$y(s) = \frac{K}{(\tau s + 1)} \frac{A}{s} = \frac{Z_1}{(\tau s + 1)} + \frac{Z_2}{s}$$

One way to get the partial fraction expansion is: first multiply each term by the denominator of term and set that term to zero:

$$\begin{aligned} (\tau s + 1) \frac{K}{(\tau s + 1)} \frac{A}{s} &= (\tau s + 1) \frac{Z_1}{(\tau s + 1)} + (\tau s + 1) \frac{Z_2}{s} \\ (\tau s + 1)|_{s=-\frac{1}{\tau}} \frac{K}{(\tau s + 1)} \frac{A}{s} &= (\tau s + 1)|_{s=-\frac{1}{\tau}} \frac{Z_1}{(\tau s + 1)} + (\tau s + 1)|_{s=-\frac{1}{\tau}} \frac{Z_2}{s} \end{aligned}$$

Some terms cancel, others don't:

$$\begin{aligned} \frac{KA}{s}|_{s=-1/\tau} &= Z_1 + 0 \\ \frac{KA}{-1/\tau} &= Z_1 + 0 \\ -KA\tau &= Z_1 \end{aligned}$$

Do this for the second term, Z_2/s

$$s \frac{K}{(\tau s + 1)} \frac{A}{s} = s \frac{Z_1}{(\tau s + 1)} + s \frac{Z_2}{s}$$

Cancel similar terms and evaluate at $s = 0$

$$\begin{aligned} \frac{K}{(\tau s + 1)} A &= s \frac{Z_1}{(\tau s + 1)} + Z_2 \\ \frac{K}{(\tau(0) + 1)} A &= 0 + Z_2 \\ KA &= Z_2 \end{aligned}$$

The result can be written:

$$y(s) = \frac{K}{(\tau s + 1)} \frac{A}{s} = \frac{Z_1}{(\tau s + 1)} + \frac{Z_2}{s}$$

Substitute in Z_1 and Z_2

$$y(s) = \frac{-KA\tau}{(\tau s + 1)} + \frac{KA}{s}$$

Simplify terms:

$$y(s) = -KA \frac{\tau}{(\tau s + 1)} + KA \frac{1}{s}$$

$$y(s) = -KA \frac{\tau}{(\tau s + \frac{1}{\tau})} \frac{1/\tau}{1/\tau} + KA \frac{1}{s}$$

$$y(s) = -KA \frac{1}{(s + \frac{1}{\tau})} + KA \frac{1}{s}$$

$$y(s) = -KA \frac{1}{(s + \frac{1}{\tau})} + KA \frac{1}{s}$$

We can invert each term in this expression. $L\{e^{-at}\}$ is $\frac{1}{s+a}$, so $L^{-1}\{\frac{1}{(s+\frac{1}{\tau})}\}$ is just $e^{-(\frac{1}{\tau})t}$. We know for the step function from 0 to 1 at time 0 the Laplace transform is $\frac{1}{s}$. The resulting solution $y(t)$ is composed of two different functions, $e^{-(\frac{1}{\tau})t}$ and a step at time 0.

$$y(s) = -KA \frac{1}{(s + \frac{1}{\tau})} + KA \frac{1}{s}$$

$$L^{-1}\{y(s)\} = L^{-1}\left\{-KA \frac{1}{(s + \frac{1}{\tau})}\right\} + L^{-1}\left\{KA \frac{1}{s}\right\}$$

Again, using the argument about constants times a function, we can pull out the KA terms.

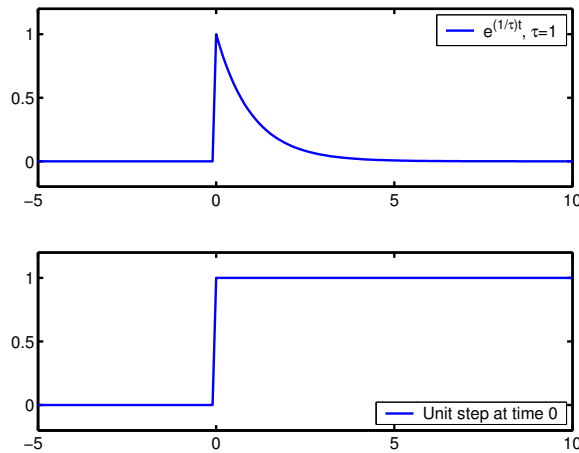
$$L^{-1}\{y(s)\} = -KA L^{-1}\left\{\frac{1}{(s + \frac{1}{\tau})}\right\} + KA L^{-1}\left\{\frac{1}{s}\right\}$$

$$y(t) = -KA e^{-(\frac{1}{\tau})t} + KA$$

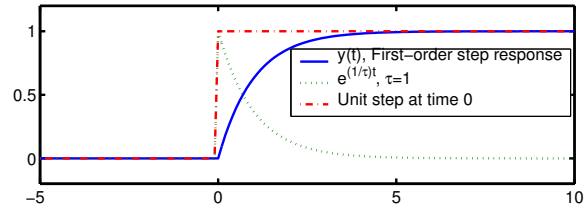
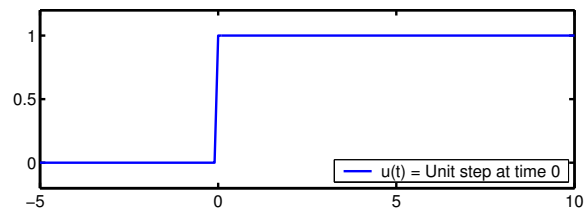
$$y(t) = KA(-e^{-(\frac{1}{\tau})t} + 1)$$

$$y(t) = KA(1 - e^{-(\frac{1}{\tau})t})$$

Laplace transforms assume everything is 0 before time 0. This function $y(t)$ only is defined for $t \geq 0$. The two separate functions that comprise $y(t)$ are shown in the following graph, $e^{-(\frac{1}{\tau})t}$ and a unit step at time zero:



Graphing the actual system (the sum of the two functions):



4.3 Deviation Variables

Lets examine a realistic First-Order system, the tank system.

$$A \frac{dh}{dt}(t) = F_i(t) - \alpha h(t)$$

Assume the flow manipulated and has units of $\frac{m^3}{s}$. The height of the tank will be measured, and the height of the tank is given in units of m . The area of the tank is $2 m^2$. For the outlet term to be consistent with the units of other terms ($\frac{m^3}{s}$), α must have units of $\frac{m^2}{s}$. Assume α has a value of $0.1 \frac{m^2}{s}$. The mass balance can be written as:

$$2 \frac{dh}{dt}(t) = F_i(t) - 0.1 h(t)$$

Now, assume that you normally operate this tank at a flow rate of entering the tank of $0.5 \frac{m^3}{s}$. This means we know the steady state flow rate into the tank, $F_{iss} = 0.5 \frac{m^3}{s}$. This also means we can figure out the steady state height of the tank from the mass balance. At steady state, $\frac{dh}{dt}(t) = 0$

$$2 \frac{dh}{dt}(t) = F_i(t) - 0.1 h(t)$$

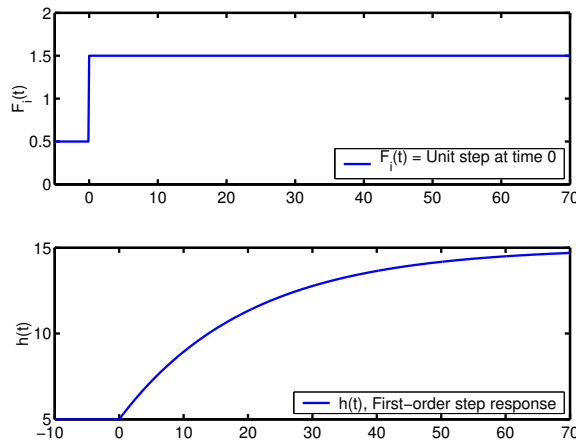
$$\frac{dh}{dt}|_{ss} = F_{iss} - 0.1 h_{ss}$$

$$0 = 0.5 \frac{m^3}{s} - 0.1 \frac{m^2}{s} h_{ss}$$

$$-0.5 \frac{m^3}{s} = -0.1 \frac{m^2}{s} h_{ss}$$

$$5 m = h_{ss}$$

So now we know h_{ss} , the steady state height of the tank. Now to make our life easier when taking Laplace transform, we put everything in **Deviation Variables**. This means we subtract the steady state from the normal functions of time. The purpose of this is to make the functions all start at a value of 0. Currently, a step response for the tank system looks like:



Using the variables in deviation form, assume $y(t) = h(t) - h_{ss}$. This means that if we start at steady state at time 0, $y(t)$ will equal 0 at the initial steady state value, $y(t)|_{t=0} = 0$. The other deviation variable can be written $u(t) = F_i(t) - F_{iss}$. This means the input $u(t)$ equals 0 at the initial starting point, $u(t)|_{t=0} = 0$. Also, taking the derivative WRT time of $y(t) = h(t) - h_{ss}$ yields

$$\frac{dy}{dt}(t) = \frac{dh}{dt}(t) - \frac{dh_{ss}}{dt}(t)$$

But h_{ss} does not change with time.

$$\begin{aligned}\frac{dy}{dt}(t) &= \frac{dh}{dt}(t) - 0 \\ \frac{dy}{dt}(t) &= \frac{dh}{dt}\end{aligned}$$

The dynamic mass balance is written as:

$$2\frac{dh}{dt}(t) = F_i(t) - 0.1 h(t)$$

The steady state mass balance is written as:

$$0 = F_{iss} - h_{ss}$$

Subtracting the steady state mass balance from the dynamic mass balance gives:

$$\begin{aligned}2\frac{dh}{dt}(t) - 0 &= F_i(t) - F_{iss} - 0.1 h(t) - (-h_{ss}) \\ 2\frac{dh}{dt}(t) &= (F_i(t) - F_{iss}) - (0.1 h(t) - h_{ss})\end{aligned}$$

And replacing what we can with deviation variables:

$$2\frac{dy}{dt}(t) = u(t) - 0.1 y(t)$$

To put this in the “traditional” $\tau \frac{dy}{dt} + y = Ku$ form, divide by 0.1.

$$\begin{aligned}\frac{2}{0.1} \frac{dy}{dt}(t) &= \frac{1}{0.1} u(t) - 1 y(t) \\ 20 \frac{dy}{dt}(t) + y(t) &= 10 u(t)\end{aligned}$$

So we know that $\tau = 20$ and $K = 10$ for this process.

Now, you can easily take the Laplace transform of this dynamic model.

$$\begin{aligned}L \left\{ 20 \frac{dy}{dt}(t) \right\} + L \{y(t)\} &= L \{10 u(t)\} \\ 20 L \left\{ \frac{dy}{dt}(t) \right\} + L \{y(t)\} &= 10 L \{u(t)\}\end{aligned}$$

$$20 (sy(s) - y(t)|_{t=0}) + y(s) = 10 u(s)$$

Since we put everything in deviation variables, $y(t)|_{t=0}$ is now 0.

$$20 (sy(s) - 0) + y(s) = 10 u(s)$$

$$20 sy(s) + y(s) = 10 u(s)$$

Solving for $y(s)$:

$$20 sy(s) + y(s) = 10 u(s)$$

$$(20 s + 1) y(s) = 10 u(s)$$

$$y(s) = \frac{10}{(20 s + 1)} u(s)$$

Again, you see this in the form $\frac{K}{\tau s + 1}$. We want to get the expression for $y(s)$ as a function of s , not a function of s and $u(s)$. We know the value for $u(t)$. In the original variables, $F_i(t)$ changed from 0.5 to 1.5 at time $t=0$. We do not know the Laplace transform for a step from 0.5 to 1.5 at time $t = 0$. In deviation variables, $u(t)$ changes from a value of 0 to a value of 1 at time $t = 0$. We know the Laplace transform of a step function from 0 to 1 at time $t = 0$. This value is $u(s) = \frac{1}{s}$

$$y(s) = \frac{10}{(20 s + 1)} \frac{1}{s}$$

Using our partial fraction expansion:

$$y(s) = \frac{-200}{(20 s + 1)} + \frac{10}{s}$$

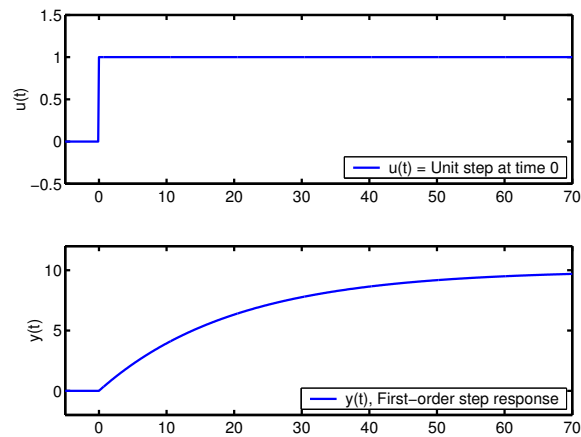
$$y(s) = \frac{-200}{(20 s + 1)} \frac{1/20}{1/20} + \frac{10}{s}$$

$$y(s) = \frac{-10}{(s + \frac{1}{20})} + \frac{10}{s}$$

$$y(s) = 10 \left(\frac{1}{s} - \frac{1}{(s + \frac{1}{20})} \right)$$

$$y(t) = 10 \left(1 - e^{-(\frac{1}{20})t} \right)$$

This expression for $y(t)$ can be plotted. Note that $y(t)$ and $u(t)$ start at zero.



What value does the response take when $t = \tau$? In this case, $\tau = 20$.

$$y(t|_{t=20}) = 10 \left(1 - e^{-\left(\frac{1}{20}\right)20} \right)$$

$$y(t|_{t=20}) = 10 (1 - e^{-1})$$

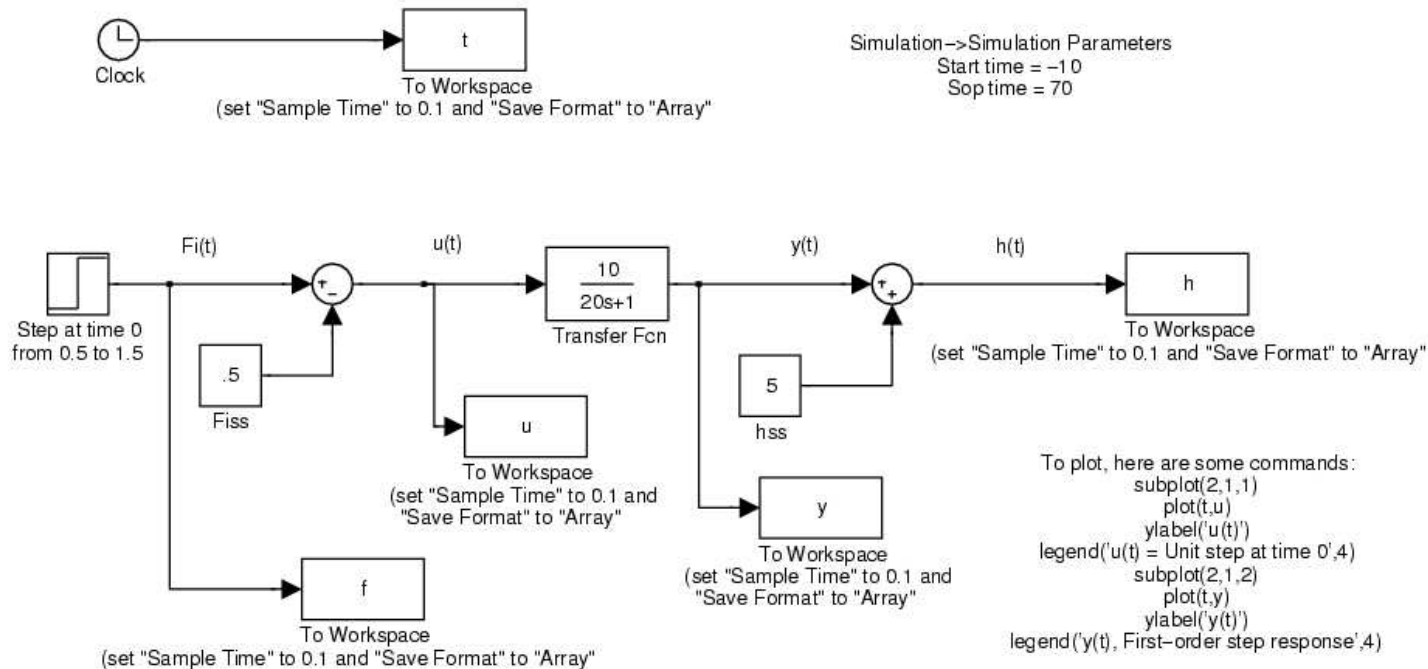
$$y(t|_{t=20}) = 10 (1 - 0.3678)$$

$$y(t|_{t=20}) = 10 (0.6321)$$

$$y(t|_{t=20}) = 6.32$$

So at time $t = \tau$ the response is 6.32, or 63% of the final value of 10.

This can also be simulated in Simulink:



After running the simulation, the results will be put in vectors in the Matlab workspace. These vectors are named (in this example) t , y , u , h , and f . Note that the step occurs at time $t = 0$, so you should start the simulation at time $t = -10$. Also note that the “To Workspace” blocks must have the “Save Format” set to “Array”.

The following plotting command will let you plot $u(t)$ and $y(t)$ on the same figure:

```
subplot(2,1,1)
plot(t,u)
ylabel('u(t)')
legend('u(t) = Unit step at time 0',4)
subplot(2,1,2) plot(t,y)
ylabel('y(t)')
legend('y(t), First-order step response',4)
```


Chapter 5

Basic Procedures for Common Problems

5.1 Steady State Multivariable Modeling and Control

1. Determine what variables are available to manipulate (inputs, Δu) and what variables are available to measure (outputs, Δy)
2. Note how many input and output variables you have.
3. Start to write equations for the **output variables**. This means write something in the form:

$$\begin{aligned}\Delta y_1 &= ??? \\ \Delta y_2 &= ??? \\ &\vdots \\ \Delta y_n &= ???\end{aligned}$$

4. Read through the problem and establish relationships between individual inputs (Δu_i) and individual outputs (Δy_j). The relationships generally represent the **gain** of the individual input output relationship, for example $\Delta y_j = K \Delta u_i$. For example: “Changing input 1 by 2% decreases output 1 by 5” means $\Delta u = 2\%$ and $\Delta y = -5$ and

$$-5 = K2$$

Or $K = -5/2$ and $\Delta y_1 = -2.5\Delta u_1$.

5. Put all of the relationships into the equations. Keep reading through the word expression until you relate all specified inputs and outputs:

$$\begin{aligned}\Delta y_1 &= -2.5\Delta u_1 + ??? \\ \Delta y_2 &= 4 ??? \\ &\vdots \\ \Delta y_n &= ???\end{aligned}$$

- Write out the equations with all input variable in every equation, even if they have a 0 coefficient.

$$\begin{aligned}\Delta y_1 &= -2.5\Delta u_1 + 0\Delta u_2 + 3\Delta u_3 \\ \Delta y_2 &= 0\Delta u_1 + 4\Delta u_2 + 1\Delta u_3 \\ \Delta y_3 &= 5\Delta u_1 + 10\Delta u_2 + 2\Delta u_3\end{aligned}$$

- Realize that this can be put in the form:

$$\underline{\Delta y} = \underline{K} \underline{\Delta u}$$

5.2 Dynamic Modeling

- Try to figure out what is changing with time. Try to figure out what are manipulated inputs ($u_i(t)$), what are disturbances ($d_i(t)$) and what are measurements ($y_i(t)$).
- Start to write dynamic mass and energy balances for the items that are changing.
- Note the accumulation term
 - Changing volume: $V(t) = Ah(t) \rightarrow A \frac{dh}{dt}(t)$
 - Changing amount of species in a tank: $VC_A(t) \rightarrow V \frac{dC_A}{dt}(t)$
 - Changing temperature in a tank: $V\rho C_p(T(t) - T^*) \rightarrow V\rho C_p \frac{dT}{dt}(t)$
- Don't forget reaction terms for reacting systems. $Vr(t)$ where $r(t)$ is the reaction rate, usually in the form $r(t) = kC_A(t)$ (or more complex).
- Write your equations and check units.

5.3 State Space

- Identify $\underline{x}$ as the values that are changing with time in your accumulation term.
- Identify your manipulated inputs $\underline{u}$.
- Identify your measurement equations. Your measurements should be expressed as functions of the states and inputs.
- Write your dynamic equations, including terms for every state and input (with 0 coefficient if necessary).
- Reorder the terms in you dynamic equations such that states come first in order, then inputs. For example:

$$\frac{dx_1}{dt} = 2x_1 + 3x_2 + 0x_3 + 2u_1 + 5u_2$$

- Put the dynamic equations in the form

$$\underline{\dot{x}} = \underline{A}\underline{x} + \underline{B}u$$

- Write your measurement equations, including terms for every state and input (with 0 coefficient if necessary).
- Put your measurement equations in the form:

$$\underline{y} = \underline{C}\underline{x} + \underline{D}u$$

5.4 Laplace Transform of Dynamic Equations

- If your steady state values are not all = 0, take your dynamic model equations and establish the steady state values for you inputs, states, and outputs. This is accomplished by solving for unknowns with the accumulation terms = 0.
- If your equations are nonlinear, **linearize** your equations.

$$A \frac{dh}{dt}(t) = F_{in}(t) - \sqrt{h(t)}$$

Here, $\sqrt{h(t)}$ is nonlinear. Near steady state, it can be approximated as

$$\sqrt{h(t)} \simeq \sqrt{h_{ss}} + \frac{1}{2}h_{ss}^{-\frac{1}{2}}(h(t) - h_{ss})$$

such that

$$A \frac{dh}{dt}(t) = F_{in}(t) - \left(\sqrt{h_{ss}} + \frac{1}{2}h_{ss}^{-\frac{1}{2}}(h(t) - h_{ss}) \right)$$

- Subtract the steady state model equations from the dynamic model equations to put everything in **deviation variables**. For example, $y(t) = h(t) - h_{ss}$ and $u(t) = F_{in}(t) - F_{inss}$.
 - Remember to express the accumulation term with your deviation variables. For $y(t) = h(t) - h_{ss}$, taking the derivative, $\frac{dy}{dt}(t) = \frac{dh}{dt}(t)$ because h_{ss} is constant.
- Express your dynamic problem using deviation variables $u(t)$, $y(t)$, $d(t)$. These functions of time should = 0 at time $t = 0$.
- Take the Laplace transform of your system.
- Solve algebraically to get in the form

$$y(s) = g(s) u(s)$$

or

$$\frac{y(s)}{u(s)} = g(s)$$

7. If you have disturbances and inputs, your model can look like

$$y(s) = g(s) u(s) + g_d(s) d(s)$$

Note that to get $g(s)$ you can assume $d(s) = 0$ then solve for $g(s)$. To get $g_d(s)$ you assume $u(s) = 0$ and solve for $g_d(s)$.

8. If you multiple inputs inputs, your model can look like

$$y(s) = g_1(s) u_1(s) + g_2(s) u_2(s)$$

9. If you have multiple inputs and multiple measurements, your model can look like

$$\begin{aligned} y_1(s) &= g_{11}(s) u_1(s) + g_{12}(s) u_2(s) \\ y_2(s) &= g_{21}(s) u_1(s) + g_{22}(s) u_2(s) \end{aligned}$$

10. Given the input as a function of time $u(t)$ (or input and disturbances) you can determine $u(s)$ (or $u(s)$ and $d(s)$).
11. Plug in to get an expression for $y(s)$ in terms of the variable s

5.5 Laplace of Complex Functions

1. You should be familiar with basic functions of time (step, impulse, ramp, exponential decay, sinusoid).
2. If the function is not 0 for $t < 0$ you should put the function in deviation variables. For example, a step in $F_{in}(t)$ at time 0 from 2 to 3 can be expressed as a unit step in $u(t)$ at time 0 with $u(t) = F_{in}(t) - F_{inss}$
3. You should be able to express the complex function as a single function of time. Multiply by the Heaviside function if needed. For a function that ramps from 0 with a slope of 2 until time 10 settling out at a value of 20, this can be expressed as

$$f(t) = 2t H(t) + (-2)(t - 10) H(t - 10)$$

4. Sketch the individual terms in your function as functions of time, then add them together to check your formulation. You can plug in numbers to check your function.
5. For each term, shift it in time such that the “event” occurs at time zero and determine the Laplace transform. Use the time shift operator if necessary to express the function as some $f(s)$. For the example:

$$f(s) = \frac{2}{s^2} + \frac{-2}{s^2} e^{-20s}$$

5.6 Solving for $y(t)$

1. Establish $y(s)$ as a function of s . (Develop dynamic model, take Laplace of model, and determine $u(s)$ and $d(s)$ if needed)
2. Your response may be in the form

$$y(s) = \frac{N_1(s)}{D_1(s)} + \frac{N_2(s)}{D_2(s)}e^{-\alpha s} + \dots + \frac{N_3(s)}{D_3(s)}e^{-\beta s}$$

This expression with multiple terms will be treated as multiple different responses, each shifted in time.

3. If you have a time delay, $e^{-\alpha s}$, ignore it for now.
4. Take a term from $y(s)$ and determine the **poles**, the roots of $D_i(s)$.
5. Perform a **Partial Fraction Expansion** on the term. For expressions with unique poles p_i the result looks like:

$$\frac{N_1(s)}{D_1(s)} = \frac{Z_1}{(s - p_1)} + \frac{Z_2}{(s - p_2)} + \dots + \frac{Z_n}{(s - p_n)}$$

For non-unique poles or imaginary roots, check the Appendix. Non-unique Poles will result in

$$\frac{Z_1}{(s - p_1)} + \frac{Z_2 s}{(s - p_1)} + \frac{Z_3 s^2}{(s - p_1)}$$

while imaginary roots result in sin or cosine in your $y(t)$

6. Now you should be able to determine the inverse Laplace transform of each expression to yield a function of time, $y_1(t)$.

$$y_1(t) = Z_1 e^{-p_1 t} + Z_2 e^{-p_2 t} + \dots + Z_n e^{-p_n t}$$

7. If you had a time delay in your term, shift the response by the time delay:

$$y_1(t) = (Z_1 e^{-p_1(t-\alpha)} + Z_2 e^{-p_2(t-\alpha)} + \dots + Z_n e^{-p_n(t-\alpha)}) H(t - \alpha)$$

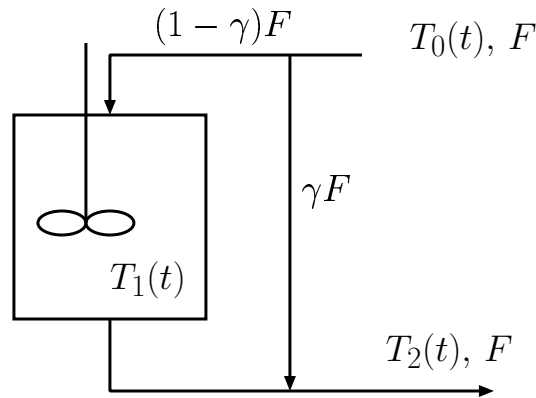
8. Do this procedure for all your terms in the original $y(s)$
9. Add up all $y_i(t)$ to get $y(t)$

Chapter 6

Lead-Lag

Objective:

A constant volume, constant flowrate mixer is used in the configuration below. Determine the unit step response for the outlet temperature $T_2(t)$.



6.1 Modeling Example System

1. Dynamic Model - Develop an energy balance for the mixing tank:

$$\frac{d(\rho V C_p (T_1(t) - T^*))}{dt} = \gamma F \rho C_p (T_o(t) - T^*) - \gamma F \rho C_p (T_1(t) - T^*)$$

$$\rho V C_p \frac{d(T_1(t) - T^*)}{dt} = \gamma F \rho C_p (T_o(t) - T^*) - \gamma F \rho C_p (T_1(t) - T^*)$$

$$\frac{V}{\gamma F} \frac{d(T_1(t) - T^*)}{dt} = (T_o(t) - T^*) - (T_1(t) - T^*)$$

2. Determine an energy balance on the mixing point:

$$0 = (1 - \gamma)F \rho C_p (T_o(t) - T^*) + \gamma F \rho C_p (T_1(t) - T^*) - F \rho C_p (T_2(t) - T^*)$$

$$0 = (1 - \gamma) (T_o(t) - T^*) + \gamma (T_1(t) - T^*) - (T_2(t) - T^*)$$

3. Put your dynamic (and steady state) equations into deviation variables. In this case, we will use the following deviation variables: $u(t) = T_o(t) - T^*$, $x(t) = T_1(t) - T^*$, and $y(t) = T_2(t) - T^*$.

$$\frac{V}{\gamma F} \frac{dx(t)}{dt} = u(t) - x(t)$$

$$0 = (1 - \gamma) u(t) + \gamma x(t) - y(t)$$

4. Take the Laplace transform of the equations:

$$\frac{V}{\gamma F} (sx(s) - x(t=0)) = u(s) - x(s)$$

$$0 = (1 - \gamma) u(s) + \gamma x(s) - y(s)$$

Because of the deviation variables, $x(t=0) = 0$

$$\frac{V}{\gamma F} (sx(s)) = u(s) - x(s)$$

Rearranging the mixing tank equation:

$$x(s) = \frac{1}{\frac{V}{\gamma F} s + 1} u(s)$$

5. We need the relationship between $u(t)$ and $y(t)$. Substitute the mixing tank equation into the mixing point equation:

$$0 = (1 - \gamma) u(s) + \frac{\gamma}{\frac{V}{\gamma F} s + 1} u(s) - y(s)$$

Rearrange to get in the form $y(s) = g(s)u(s)$

$$0 = (1 - \gamma) u(s) + \frac{\gamma}{\frac{V}{\gamma F} s + 1} u(s) - y(s)$$

$$y(s) = (1 - \gamma) u(s) + \frac{\gamma}{\frac{V}{\gamma F} s + 1} u(s)$$

$$y(s) = (1 - \gamma) u(s) + \frac{\gamma}{\frac{V}{\gamma F} s + 1} u(s)$$

$$y(s) = (1 - \gamma) u(s) \left(\frac{\frac{V}{\gamma F} s + 1}{\frac{V}{\gamma F} s + 1} \right) + \frac{\gamma}{\frac{V}{\gamma F} s + 1} u(s)$$

$$y(s) = \frac{(1 - \gamma) u(s) \left(\frac{V}{\gamma F} s + 1 \right) + \gamma u(s)}{\frac{V}{\gamma F} s + 1}$$

$$y(s) = \frac{u(s) \left((1 - \gamma) \frac{V}{\gamma F} s + (1 - \gamma) \right) + \gamma u(s)}{\frac{V}{\gamma F} s + 1}$$

$$y(s) = \frac{u(s) \left((1 - \gamma) \frac{V}{\gamma F} s + (1 - \gamma + \gamma) \right)}{\frac{V}{\gamma F} s + 1}$$

$$y(s) = \frac{u(s) \left((1 - \gamma) \frac{V}{\gamma F} s + 1 \right)}{\frac{V}{\gamma F} s + 1}$$

$$\frac{y(s)}{u(s)} = \frac{\left(\frac{V(1-\gamma)}{\gamma F} s + 1 \right)}{\frac{V}{\gamma F} s + 1}$$

For $\gamma = 1$ this reduces to (a first order system):

$$\frac{y(s)}{u(s)} = \frac{1}{\frac{V}{\gamma F} s + 1}$$

For $\gamma = 0$ this reduces to a pure gain system. The original equation

$$\frac{y(s)}{u(s)} = \frac{\left(\frac{V(1-\gamma)}{\gamma F} s + 1 \right)}{\frac{V}{\gamma F} s + 1}$$

is in the form

$$\frac{y(s)}{u(s)} = \frac{K (\xi s + 1)}{\tau s + 1}$$

with $K = 1$, $\tau = \frac{V}{\gamma F}$, and $\xi = \frac{V(1-\gamma)}{\gamma F}$.

If we want this in the form:

$$\begin{aligned} \frac{K (\xi s + 1)}{\tau s + 1} &= A_0 + \frac{A_1}{\tau s + 1} \\ \frac{K (\xi s + 1)}{\tau s + 1} &= \frac{A_0 \tau s + A_0}{\tau s + 1} + \frac{A_1}{\tau s + 1} \\ \frac{K (\xi s + 1)}{\tau s + 1} &= \frac{A_0 \tau s + A_0 + A_1}{\tau s + 1} \\ \frac{K \xi s + K}{\tau s + 1} &= \frac{A_0 \tau s + (A_0 + A_1)}{\tau s + 1} \end{aligned}$$

$$K \xi = A_0 \tau$$

$$K = A_0 + A_1$$

$$\begin{aligned}\frac{K\xi}{\tau} &= A_0 \\ K &= \frac{K\xi}{\tau} + A_1\end{aligned}$$

$$\begin{aligned}A_1 &= K - \frac{K\xi}{\tau} \\ A_1 &= K \left(1 - \frac{\xi}{\tau}\right) \\ A_0 &= K \frac{\xi}{\tau}\end{aligned}$$

Now, letting $\rho = \frac{\xi}{\tau}$

$$\begin{aligned}A_0 &= K\rho \\ A_1 &= K(1 - \rho)\end{aligned}$$

$$\frac{K(\xi s + 1)}{\tau s + 1} = K\rho + \frac{K(1 - \rho)}{\tau s + 1}$$

This means the lead lag transfer function is really just two systems in parallel, a pure gain system and a first order system. The value ρ can be seen as a weighting value.

Back to the problem, we wanted step response. This means that $u(s) = \frac{1}{s}$

$$y(s) = \frac{K(\xi s + 1)}{\tau s + 1} \frac{1}{s}$$

Using partial fraction expansion, we need to break this down to

$$y(s) = \frac{K(\xi s + 1)}{\tau s + 1} \frac{1}{s} = \frac{Z_1}{\tau s + 1} + \frac{Z_2}{s}$$

Multiply by $\tau s + 1$ and set $s = -\frac{1}{\tau}$ to get Z_1 .

$$\begin{aligned}Z_1 &= \frac{K(\xi s + 1)}{s} \Big|_{s=-\frac{1}{\tau}} \\ Z_1 &= \frac{K\left(\xi - \frac{1}{\tau} + 1\right)}{-\frac{1}{\tau}} \\ Z_1 &= -K\tau \left(\xi \left(-\frac{1}{\tau}\right) + 1\right)\end{aligned}$$

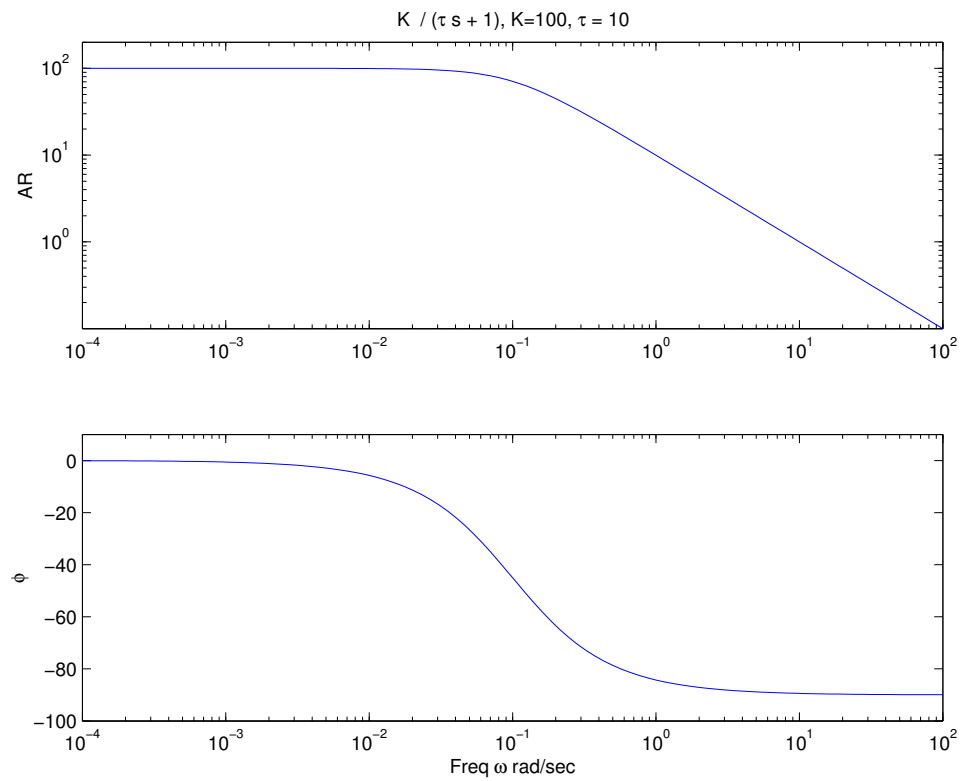
To get Z_2 , multiply by s and set $s = 0$

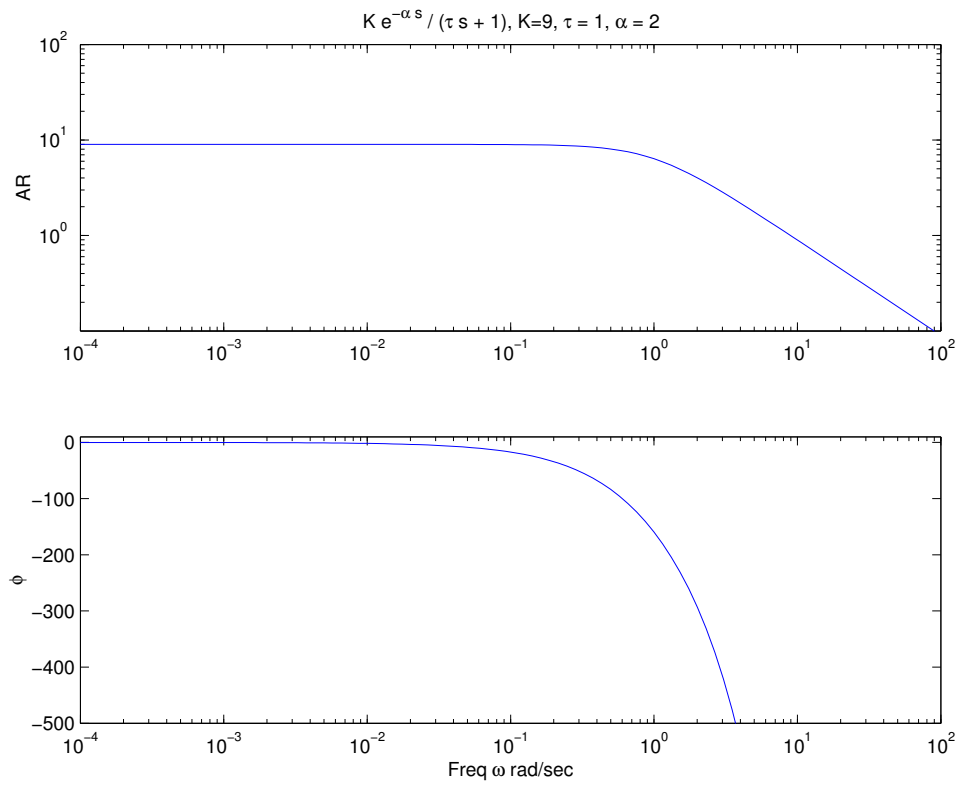
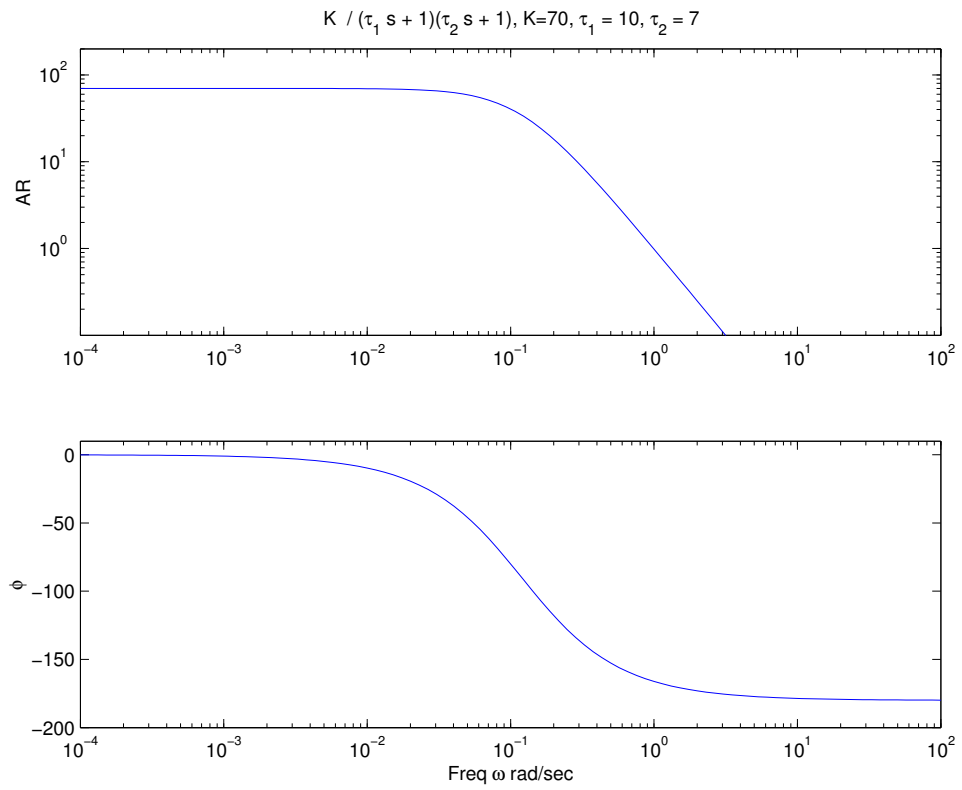
$$Z_2 = \frac{1}{1} = 1$$

Chapter 7

Frequency Analysis

7.1 Bode Plots of Simple Systems





7.2 Derivations of Frequency Response for Simple Systems

7.2.1 First-Order System

$$g(s) = \frac{K}{\tau s + 1}$$

$$g(j\omega) = \frac{K}{\tau j\omega + 1}$$

$$g(j\omega) = \frac{K}{1 + \tau\omega j}$$

$$g(j\omega) = \frac{K}{1 + \tau\omega j} \frac{1 - \tau\omega j}{1 - \tau\omega j}$$

$$g(j\omega) = \frac{K(1 - \tau\omega j)}{(1 + \tau\omega j)(1 - \tau\omega j)}$$

$$g(j\omega) = \frac{K - K\tau\omega j}{1 + \tau^2\omega^2 j^2}$$

$$g(j\omega) = \frac{K - K\tau\omega j}{1 + \tau^2\omega^2(-1)}$$

$$g(j\omega) = \frac{K - K\tau\omega j}{1 - \tau^2\omega^2}$$

$$g(j\omega) = \frac{K}{1 - \tau^2\omega^2} - \frac{K\tau\omega}{1 - \tau^2\omega^2}j$$

$$AR(\omega) = |g(j\omega)| = \sqrt{\left(\frac{K}{1 - \tau^2\omega^2}\right)^2 + \left(\frac{-K\tau\omega}{1 - \tau^2\omega^2}\right)^2}$$

$$AR(\omega) = |g(j\omega)| = \sqrt{\frac{K^2 + (-K\tau\omega)^2}{(1 - \tau^2\omega^2)^2}}$$

$$AR(\omega) = |g(j\omega)| = \sqrt{\frac{K^2 + K^2\tau^2\omega^2}{(1 - \tau^2\omega^2)^2}}$$

$$AR(\omega) = |g(j\omega)| = \sqrt{\frac{K^2(1 + \tau^2\omega^2)}{(1 - \tau^2\omega^2)^2}}$$

$$AR(\omega) = |g(j\omega)| = K \sqrt{\frac{(1 + \tau^2\omega^2)}{(1 - \tau^2\omega^2)^2}}$$

$$AR(\omega) = |g(j\omega)| = K \sqrt{\frac{1}{(1 - \tau^2\omega^2)}}$$

For phase angle as a function of frequency ω

$$\phi(\omega) = \angle g(j\omega) = \arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{\left(\frac{-K\tau\omega}{1-\tau^2\omega^2}\right)}{\left(\frac{K}{1-\tau^2\omega^2}\right)}\right)$$

$$\phi(\omega) = \angle g(j\omega) = \arctan(-\tau\omega)$$

7.2.2 Second-Order System

$$g(s) = \frac{K}{\tau^2 s^2 + 2\tau\zeta s + 1}$$

$$g(j\omega) = \frac{K}{\tau^2 (j\omega)^2 + 2\tau\zeta(j\omega) + 1}$$

$$g(j\omega) = \frac{K}{\tau^2 (-1)\omega^2 + 2\tau\zeta j\omega + 1}$$

$$g(j\omega) = \frac{K}{1 - \tau^2\omega^2 + 2\tau\zeta\omega j}$$

$$g(j\omega) = \frac{K}{(1 - \tau^2\omega^2) + 2\tau\zeta\omega j} \frac{(1 - \tau^2\omega^2) - 2\tau\zeta\omega j}{(1 - \tau^2\omega^2) - 2\tau\zeta\omega j}$$

$$g(j\omega) = \frac{K((1 - \tau^2\omega^2) - 2\tau\zeta\omega j)}{(1 - \tau^2\omega^2)^2 - (2\tau\zeta\omega j)^2}$$

$$g(j\omega) = \frac{K((1 - \tau^2\omega^2) - 2\tau\zeta\omega j)}{(1 - \tau^2\omega^2)^2 - (-1)(2\tau\zeta\omega)^2}$$

$$g(j\omega) = \frac{K((1 - \tau^2\omega^2) - 2\tau\zeta\omega j)}{(1 - \tau^2\omega^2)^2 + (2\tau\zeta\omega)^2}$$

$$g(j\omega) = \frac{K(1 - \tau^2\omega^2)}{(1 - \tau^2\omega^2)^2 + (2\tau\zeta\omega)^2} - \frac{K2\tau\zeta\omega}{(1 - \tau^2\omega^2)^2 + (2\tau\zeta\omega)^2} j$$

$$AR(\omega) = |g(j\omega)| = \sqrt{\left(\frac{K(1 - \tau^2\omega^2)}{(1 - \tau^2\omega^2)^2 + (2\tau\zeta\omega)^2}\right)^2 + \left(\frac{-K2\tau\zeta\omega}{(1 - \tau^2\omega^2)^2 + (2\tau\zeta\omega)^2}\right)^2}$$

$$AR(\omega) = |g(j\omega)| = \sqrt{\frac{K^2(1 - \tau^2\omega^2)^2 + (-K2\tau\zeta\omega)^2}{((1 - \tau^2\omega^2)^2 + (2\tau\zeta\omega)^2)^2}}$$

$$AR(\omega) = |g(j\omega)| = \sqrt{\frac{K^2(1 - \tau^2\omega^2)^2 + K^2(2\tau\zeta\omega)^2}{((1 - \tau^2\omega^2)^2 + (2\tau\zeta\omega)^2)^2}}$$

$$AR(\omega) = |g(j\omega)| = \sqrt{\frac{K^2((1 - \tau^2\omega^2)^2 + (2\tau\zeta\omega)^2)}{((1 - \tau^2\omega^2)^2 + (2\tau\zeta\omega)^2)^2}}$$

$$AR(\omega) = |g(j\omega)| = K \sqrt{\frac{((1 - \tau^2\omega^2)^2 + (2\tau\zeta\omega)^2)}{((1 - \tau^2\omega^2)^2 + (2\tau\zeta\omega)^2)^2}}$$

$$AR(\omega) = |g(j\omega)| = K \sqrt{\frac{1}{((1 - \tau^2\omega^2)^2 + (2\tau\zeta\omega)^2)}}$$

For phase angle as a function of frequency ω

$$\phi(\omega) = \angle g(j\omega) = \arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{-\frac{K2\tau\zeta\omega}{(1-\tau^2\omega^2)^2+(2\tau\zeta\omega)^2}}{\frac{K(1-\tau^2\omega^2)}{(1-\tau^2\omega^2)^2+(2\tau\zeta\omega)^2}}\right)$$

$$\phi(\omega) = \angle g(j\omega) = \arctan\left(\frac{-2\tau\zeta\omega}{(1 - \tau^2\omega^2)}\right)$$

$$\phi(\omega) = \angle g(j\omega) = \arctan(-\tau\omega)$$

7.2.3 Time Delay System

$$g(s) = e^{-\alpha s}$$

$$g(j\omega) = e^{-\alpha j\omega}$$

$$g(j\omega) = e^{-\alpha\omega j}$$

Using the Euler Identity:

$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

$$g(j\omega) = e^{(-\alpha\omega)j} = \cos(-\alpha\omega) + j \sin(-\alpha\omega)$$

$$g(j\omega) = \cos(-\alpha\omega) + \sin(-\alpha\omega) j$$

$$AR(\omega) = |g(j\omega)| = \sqrt{(\cos(-\alpha\omega))^2 + (\sin(-\alpha\omega))^2}$$

$$AR(\omega) = |g(j\omega)| = \sqrt{1} = 1$$

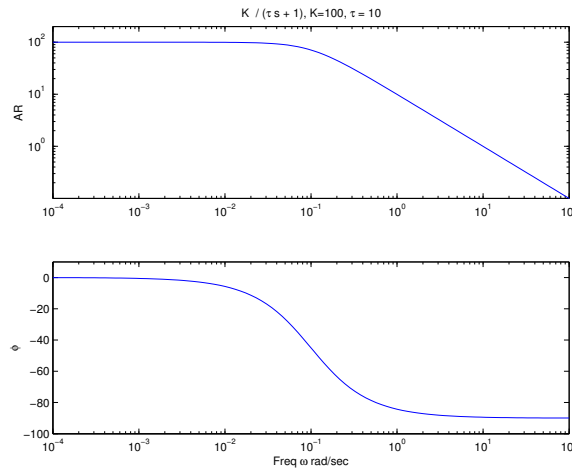
$$\phi(\omega) = \angle g(j\omega) = \arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{\sin(-\alpha\omega)}{\cos(-\alpha\omega)}\right)$$

$$\phi(\omega) = \angle g(j\omega) = \arctan(\tan(-\alpha\omega))$$

$$\phi(\omega) = \angle g(j\omega) = -\alpha\omega$$

7.3 Frequency Response Questions

1. The Bode Plot for a first order system is given below. Identify the transfer function for the system.



2. Sketch the Bode plot for the following transfer function. Label any distinguishing characteristics.

$$g(s) = \frac{100e^{-2s}}{(10s + 1)}$$

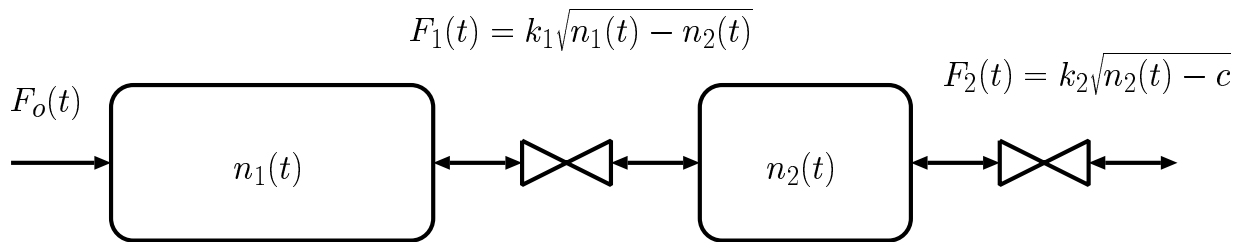
3. You are in charge of operating the sludge furnace at the local Ideal Gas company plant. You must design a holding tank with limited level variation, given that the supply flow of sludge varies beyond your control. The flow rate from the upstream process varies with a period of 45 min and an amplitude of $\pm 1 \frac{m^3}{hr}$. Your goal is to calculate the cross sectional area of a buffer tank that will vary in height by $\pm 0.1 m$. The flow rate from the tank is given as $F = kh$ where $k = 1 \frac{m^2}{hr}$.

- What is the frequency of upstream oscillation in $\frac{rad}{hr}$?
 - What is the transfer function for the system in the form $\frac{K}{\tau s + 1}$ relating the upstream input flow rate to the tank liquid level?
 - For this system, what is the expression for the Amplitude Ratio as a function of ω ?
 - What is the area of the tank in m^2 that will limit level variation to $\pm 0.1 m$?
4. Your boss at the Ideal Gas Company put you in charge of analyzing two tanks, each with cross sectional area of $2 m^2$. The tanks are arranged in series. The flow from tank 1 to tank 2 is $F_1 = kh_1$ and the flow from tank 2 is $F_2 = kh_2$. The flow into the first tank is known to vary with a frequency of $.5 \frac{rad}{hr}$. You are told that $k = 2 \frac{m^2}{hr}$.

- What is the transfer function for the process relating the flow into tank 1 to the flow out of tank 2?
- For this system, what is the expression for the Amplitude Ratio and Phase Angle as a function of ω ?
- What is amplitude of the variation in the flow out of tank 2 as a function of ω ?
- For a frequency of oscillation of $10 \frac{rad}{hr}$ What is amplitude of the variation in the flow out of tank 2?

Chapter 8

Tank Case Study



A two tank system is arranged in series as shown in the above figure. The molar gas flow rate into tank 1 can be changed by the operator. The tanks are constant volume and isothermal. The volumetric flow of gas across a valve is usually written $F = k\sqrt{\Delta P}$. In this case, using ideal gas law $PV = nRT$ you should realize that the molar amount of gas in each tank is proportional to the pressure in the tank, $n = \frac{V}{RT} P$ where V , R , and T are constant. As a result, the molar flow rate between the two tanks across a valve can be written as $F_1(t) = k_1\sqrt{n_1(t) - n_2(t)}$ and the flow across a valve to the atmosphere can be written $F_2(t) = k_2\sqrt{n_2(t) - c}$

1. Develop a dynamic mass balance for the two tank system.
2. Linearize any nonlinear terms.
3. Develop linear dynamic approximation for the system.
4. Take the Laplace transform of your linear ordinary differential equations.
5. Derive the transfer function relating the input flow to the number of moles in tank 2.
6. Sketch the bode plot for this system.
7. Assuming a feedback controller of the form

$$g_c = K_c + \frac{K_c}{\tau_I s}$$

derive the closed-loop