Correction for Simple System Example and Notes on Laplace Transforms / Deviation Variables ECHE 550 Fall 2002

Consider a tank draining from an initial height of h_o at time $t = 0$. With no flow into the tank ($F_{in} = 0$) and $F_{out} = \alpha h(t)$ the mass balance can be written:

$$A\frac{dh}{dt}(t) = 0 - \alpha\, h(t)$$

Moving $\alpha\, h(t)$ to the left half side and dividing by α gives:

$$\frac{A}{\alpha}\,\frac{dh}{dt}(t) + h(t) = 0$$

A is the tank area (constant) and α is the proportionality constant for flow out of the tank. These parameters can be replaced by $\tau = A/\alpha$ to give the following differential equation:

$$\tau\,\frac{dy}{dt}(t) + y(t) = 0 \tag{1}$$

The initial tank height at time $t = 0$ **can be assumed to be** $y(t)|_{t=0} = y_o$. Take the Laplace transform of Equation 1:

$$L\left\{\tau\,\frac{dy}{dt}(t)\right\} + L\left\{y(t)\right\} = 0$$

$L\{y(t)\}$ is easy, $L\{y(t)\} = y(s)$ so we have:

$$L\left\{\tau\,\frac{dy}{dt}(t)\right\} + y(s) = 0$$

$L\left\{\tau\,\frac{dy}{dt}(t)\right\}$ is a bit more complex. First, you can realize that τ is constant. Convince yourself of this! The L operator on a constant times a function is the same as a constant times the Laplace of the function:

$$L\left\{c\, f(t)\right\} = \int_0^\infty c\, e^{-st}\, f(t) = c\int_0^\infty e^{-st}\, f(t) = c\, L\left\{f(t)\right\}$$

So you can take the constant value outside the L operator:

$$\tau\, L\left\{\frac{dy}{dt}(t)\right\} + y(s) = 0$$

Now, you must remember that $L\left\{\frac{df}{dt}(t)\right\}$ is just $s\, f(s) - f(t)|_{t=0}$.

$$\tau\,\left(sy(s) - y(t)|_{t=0}\right) + y(s) = 0$$

And we have initial conditions for the height of the tank, $y(t)|_{t=0} = y_o$

$$\tau\ (sy(s) - y_o) + y(s) = 0$$

Solving for $y(s)$:

$$\tau\, sy(s) - \tau\, y_o + y(s) = 0$$
$$\tau\, sy(s) + y(s) = \tau\, y_o$$
$$(\tau s + 1)\, y(s) = \tau\, y_o$$
$$y(s) = \frac{\tau\, y_o}{(\tau s + 1)}$$

Now rearrange a little bit

$$y(s) = \tau\, y_o\, \frac{1}{(\tau s + 1)}$$
$$y(s) = \tau\, y_o\, \frac{1}{(\tau s + 1)}\, \frac{1/\tau}{1/\tau}$$
$$y(s) = y_o\, \frac{1}{(s + \frac{1}{\tau})}$$

This you realize is a constant y_o times the term $\frac{1}{s+\frac{1}{\tau}}$. To get $y(t)$ you must use the inverse Laplace transform, L^{-1}.

$$L^{-1}\left\{y(s)\right\} = L^{-1}\left\{y_o\, \frac{1}{(s + \frac{1}{\tau})}\right\}$$

Again, y_o is a constant and can be factored out

$$L^{-1}\left\{y(s)\right\} = y_o\, L^{-1}\left\{\frac{1}{(s + \frac{1}{\tau})}\right\}$$

And we know from lecture that $L\left\{e^{-at}\right\} = \frac{1}{s+a}$, so in our case, $a = \frac{1}{\tau}$.

$$y(t) = y_o\, e^{-(\frac{1}{\tau})t}$$

This is the solution to the original differential equation! Now check your result. At time $t = 0$ your solution for $y(t)$ is $y_o\, e^{-(\frac{1}{\tau})0}$ = $y_o\, 1 = y_o$. This matches the initial conditions. The derivative of your result can also be found

$$\frac{dy}{dt}(t) = \frac{d}{dt}\left\{y_o\, e^{-(\frac{1}{\tau})t}\right\} = y_o\ -(\frac{1}{\tau})\, e^{-(\frac{1}{\tau})t}$$
$$\frac{dy}{dt}(t) = -\frac{y_o}{\tau}\, e^{-(\frac{1}{\tau})t}$$

Plug that back in the original differential EQ, along with your solution for $y(t)$:

$$\tau\, \frac{dy}{dt}(t) + y(t) = 0$$

$$\tau\, (-\frac{y_o}{\tau})\, e^{-(\frac{1}{\tau})t} + y_o\, e^{-(\frac{1}{\tau})t} = 0$$

And we know we have the solution!

NOTES on first order system modeling

The first order system model is:

$$\tau \frac{dy}{dt}(t) + y(t) = K\,u(t)$$

Taking the Laplace transform:

$$\tau\, sy(s) + \tau y(t)|_{t=0} + y(s) = K\,u(s)$$

If we assume that $y(t)|_{t=0} = 0$ this simplifies the equation to

$$\tau\, sy(s) + y(s) = K\,u(s)$$

We can then solve for $y(s)$

$$(\tau\, s + 1)y(s) = K\,u(s)$$

$$y(s) = \frac{K}{(\tau\, s + 1)}\,u(s)$$

Here, $\frac{K}{(\tau\, s+1)}$ is the process model relating $u(s)$ and $y(s)$. This is sometimes called $g(s) = \frac{K}{(\tau\, s+1)}$. Given $u(t)$ you can find $u(s)$, and given a model of your system you can find $g(s)$. Realizing that $y(s) = g(s)u(s)$ you can then find $y(t)$.

From a ***process reaction curve*** (the data for $y(t)$ and $u(t)$ given a step in the input $u(t)$) you can find the PROCESS GAIN K from the equation:

$$K = \frac{y_{fin} - y_{init}}{u_{fin} - u_{init}} = \frac{\Delta y}{\Delta u}$$

The time constant is a bit trickier. First, lets assume $u(t)$ is a step at time $t = 0$ from a value of 0 to a new value of A. The Laplace transform of the step function is:

$$u(s) = \frac{A}{s}$$

Now, we have enough information to get $y(s)$ and $y(t)$

$$y(s) = \frac{K}{(\tau\, s + 1)}\,u(s)$$

$$y(s) = \frac{K}{(\tau\, s + 1)}\,\frac{A}{s}$$

To solve this easily, we need the partial fraction expansion:

$$y(s) = \frac{K}{(\tau\, s + 1)}\,\frac{A}{s} = \frac{Z_1}{(\tau\, s + 1)} + \frac{Z_2}{s}$$

One way to get the partial fraction expansion is: first multiply each term by the denominator of term and set that term to zero:

$$(\tau\, s + 1)\,\frac{K}{(\tau\, s + 1)}\,\frac{A}{s} = (\tau\, s + 1)\,\frac{Z_1}{(\tau\, s + 1)} + (\tau\, s + 1)\,\frac{Z_2}{s}$$

$$(\tau\, s+1)|_{s=-\frac{1}{\tau}}\ \frac{K}{(\tau\, s+1)}\,\frac{A}{s} = (\tau\, s+1)|_{s=-\frac{1}{\tau}}\ \frac{Z_1}{(\tau\, s+1)} + (\tau\, s+1)|_{s=-\frac{1}{\tau}}\ \frac{Z_2}{s}$$

Some terms cancel, others don't:

$$\frac{KA}{s}|_{s=-1/\tau} = Z_1 + 0$$

$$\frac{KA}{-1/\tau} = Z_1 + 0$$

$$-KA\tau = Z_1$$

Do this for the second term, Z_2/s

$$s\frac{K}{(\tau\, s+1)}\,\frac{A}{s} = s\frac{Z_1}{(\tau\, s+1)} + s\frac{Z_2}{s}$$

Cancel similar terms and evaluate at $s = 0$

$$\frac{K}{(\tau\, s+1)}\,A = s\frac{Z_1}{(\tau\, s+1)} + Z_2$$

$$\frac{K}{(\tau\,(0)+1)}\,A = 0 + Z_2$$

$$KA = Z_2$$

The result can be written:

$$y(s) = \frac{K}{(\tau\, s+1)}\,\frac{A}{s} = \frac{Z_1}{(\tau\, s+1)} + \frac{Z_2}{s}$$

Substitute in Z_1 and Z_2

$$y(s) = \frac{-KA\tau}{(\tau\, s+1)} + \frac{KA}{s}$$

Simplify terms:

$$y(s) = -KA\,\frac{\tau}{(\tau\, s+1)} + KA\,\frac{1}{s}$$

$$y(s) = -KA\,\frac{\tau}{(\tau\, s+\frac{1}{\tau})}\,\frac{1/\tau}{1/\tau} + KA\,\frac{1}{s}$$

$$y(s) = -KA\,\frac{1}{(s+\frac{1}{\tau})} + KA\,\frac{1}{s}$$

$$y(s) = -KA\,\frac{1}{(s+\frac{1}{\tau})} + KA\,\frac{1}{s}$$

We can invert each term in this expression. $L\{e^{-at}\}$ is $\frac{1}{s+a}$, so $L^{-1}\{\frac{1}{(s+\frac{1}{\tau})}\}$ is just $e^{-(\frac{1}{\tau})\,t}$. We know for the step function from 0 to 1 at time 0 the Laplace transform is $\frac{1}{s}$. The resulting solution $y(t)$ is composed of two different functions, $e^{-(\frac{1}{\tau})\,t}$ and a step at time 0.

$$y(s) = -KA\,\frac{1}{(s+\frac{1}{\tau})} + KA\,\frac{1}{s}$$

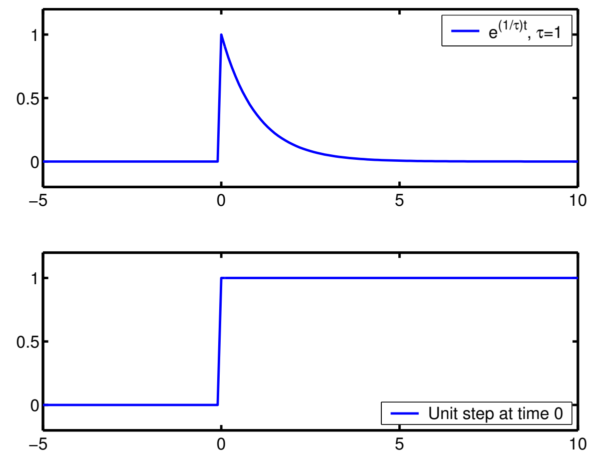
$$L^{-1}\left\{y(s)\right\} = L^{-1}\left\{-KA\,\frac{1}{(s+\frac{1}{\tau})}\right\} + L^{-1}\left\{KA\,\frac{1}{s}\right\}$$

Again, using the argument about constants times a function, we can pull out the KA terms.

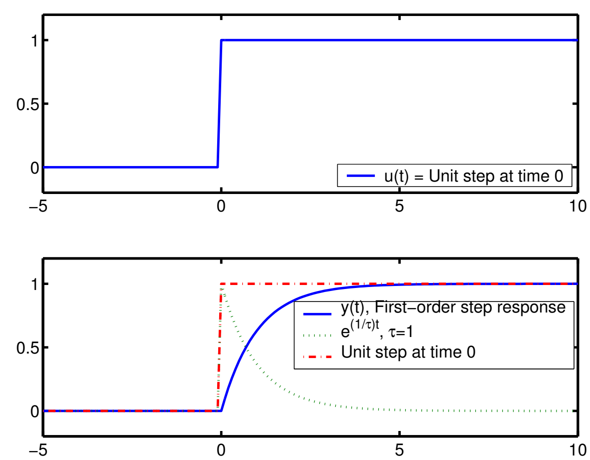
$$L^{-1}\left\{y(s)\right\} = -KA\,L^{-1}\left\{\frac{1}{(s+\frac{1}{\tau})}\right\} + KA\,L^{-1}\left\{\frac{1}{s}\right\}$$

$$y(t) = -KA\,e^{-(\frac{1}{\tau})\,t} + KA$$
$$y(t) = KA(-\,e^{-(\frac{1}{\tau})\,t} + 1)$$
$$y(t) = KA(1 - e^{-(\frac{1}{\tau})\,t})$$

Laplace transforms assume everything is 0 before time 0. This function $y(t)$ only is defined for $t \geq 0$. The two separate functions that comprise $y(t)$ are shown in the following graph, $e^{-(\frac{1}{\tau})\,t}$ and a unit step at time zero:



Graphing the actual system (the sum of the two functions):



NOTES on Deviation Variables

Lets examine a realistic First-Order system, the tank system.

$$A\frac{dh}{dt}(t) = F_i(t) - \alpha\, h(t)$$

Assume the flow manipulated and has units of $\frac{m^3}{s}$. The height of the tank will be measured, and the height of the tank is given in units of m. The area of the tank is $2\,m^2$. For the outlet term to be consistent with the units of other terms ($\frac{m^3}{s}$), α must have units of $\frac{m^2}{s}$. Assume α has a value of $0.1\,\frac{m^2}{s}$. The mass balance can be written as:

$$2\frac{dh}{dt}(t) = F_i(t) - 0.1\, h(t)$$

Now, assume that you normally operate this tank at a flow rate of entering the tank of $0.5\,\frac{m^3}{s}$. This means we know the steady state flow rate into the tank, $F_{iss} = 0.5\,\frac{m^3}{s}$. This also means we can figure out the steady state height of the tank from the mass balance. At steady state, $\frac{dh}{dt}(t) = 0$

$$2\frac{dh}{dt}(t) = F_i(t) - 0.1\, h(t)$$

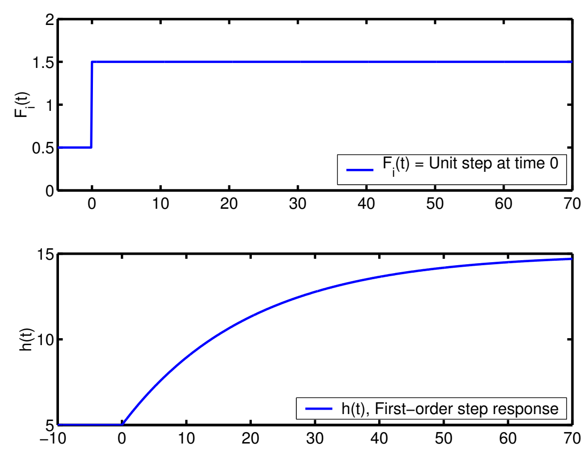
$$\frac{dh}{dt}|_{ss} = F_{iss} - 0.1\, h_{ss}$$

$$0 = 0.5\,\frac{m^3}{s} - 0.1\,\frac{m^2}{s}\,h_{ss}$$

$$-0.5\,\frac{m^3}{s} = -0.1\,\frac{m^2}{s}\,h_{ss}$$

$$5\,m = h_{ss}$$

So now we know h_{ss}, the steady state height of the tank. Now to make our life easier when taking Laplace transform, we put everything in **Deviation Variables**. This means we subtract the steady state from the normal functions of time. The purpose of this is to make the functions all start at a value of 0. Currently, a step response for the tank system looks like:



Using the variables in deviation form, assume $y(t) = h(t) - h_{ss}$. This means that if we start at steady state at time 0, $y(t)$ will equal 0 at the initial steady state value, $y(t)|_{t=0} = 0$. The other deviation variable can be written

$u(t) = F_i(t) - F_{iss}$. This means the input $u(t)$ equals 0 at the initial starting point, $u(t)|_{t=0} = 0$. Also, taking the derivative WRT time of $y(t) = h(t) - h_{ss}$ yields

$$\frac{dy}{dt}(t) = \frac{dh}{dt}(t) - \frac{dh_{ss}}{dt}(t)$$

But h_{ss} does not change with time.

$$\frac{dy}{dt}(t) = \frac{dh}{dt}(t) - 0$$

$$\frac{dy}{dt}(t) = \frac{dh}{dt}$$

The dynamic mass balance is written as:

$$2\frac{dh}{dt}(t) = F_i(t) - 0.1\,h(t)$$

The steady state mass balance is written as:

$$0 = F_{iss} - h_{ss}$$

Subtracting the steady state mass balance from the dynamic mass balance gives:

$$2\frac{dh}{dt}(t) - 0 = F_i(t) - F_{iss} - 0.1\,h(t) - (-h_{ss})$$

$$2\frac{dh}{dt}(t) = (F_i(t) - F_{iss}) - (0.1\,h(t) - h_{ss})$$

And replacing what we can with deviation variables:

$$2\frac{dy}{dt}(t) = u(t) - 0.1\,y(t)$$

To put this in the "traditional" $\tau\frac{dy}{dt} + y = Ku$ form, divide by 0.1.

$$\frac{2}{0.1}\frac{dy}{dt}(t) = \frac{1}{0.1}u(t) - 1\,y(t)$$

$$20\frac{dy}{dt}(t) + y(t) = 10\,u(t)$$

So we know that $\tau = 20$ and $K = 10$ for this process.

Now, you can easily take the Laplace transform of this dynamic model.

$$L\left\{20\,\frac{dy}{dt}(t)\right\} + L\left\{y(t)\right\} = L\left\{10\,u(t)\right\}$$

$$20\,L\left\{\frac{dy}{dt}(t)\right\} + L\left\{y(t)\right\} = 10\,L\left\{u(t)\right\}$$

$$20\,\left(sy(s) - y(t)|_{t=0}\right) + y(s) = 10\,u(s)$$

Since we put everything in deviation variables, $y(t)|_{t=0}$ is now 0.

$$20\ (sy(s)-0)+y(s)=10\,u(s)$$
$$20\,sy(s)+y(s)=10\,u(s)$$

Solving for $y(s)$:

$$20\,sy(s)+y(s)=10\,u(s)$$
$$(20\,s+1)\,y(s)=10\,u(s)$$
$$y(s)=\frac{10}{(20\,s+1)}\,u(s)$$

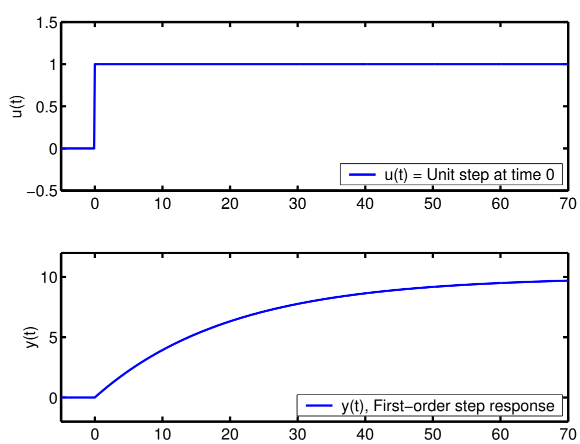
Again, you see this in the form $\frac{K}{\tau s+1}$. We want to get the expression for $y(s)$ as a function of s, not a function of s and $u(s)$. We know the value for $u(t)$. In the original variables, $F_i(t)$ changed from 0.5 to 1.5 at time t=0. We do not know the Laplace transform for a step from 0.5 to 1.5 at time $t=0$. In deviation variables, $u(t)$ changes from a value of 0 to a value of 1 at time $t=0$. We know the Laplace transform of a step function from 0 to 1 at time $t=0$. This value is $u(s)=\frac{1}{s}$

$$y(s)=\frac{10}{(20\,s+1)}\,\frac{1}{s}$$

Using our partial fraction expansion:

$$y(s)=\frac{-200}{(20\,s+1)}+\frac{10}{s}$$
$$y(s)=\frac{-200}{(20\,s+1)}\frac{1/20}{1/20}+\frac{10}{s}$$
$$y(s)=\frac{-10}{(s+\frac{1}{20})}+\frac{10}{s}$$
$$y(s)=10\left(\frac{1}{s}-\frac{1}{(s+\frac{1}{20})}\right)$$
$$y(t)=10\left(1-e^{-(\frac{1}{20})t}\right)$$

This expression for $y(t)$ can be plotted. Note that $y(t)$ and $u(t)$ start at zero.



What value does the response take when $t = \tau$? In this case, $\tau = 20$.

$$y(t|_{t=20}) = 10\left(1 - e^{-(\frac{1}{20})20}\right)$$

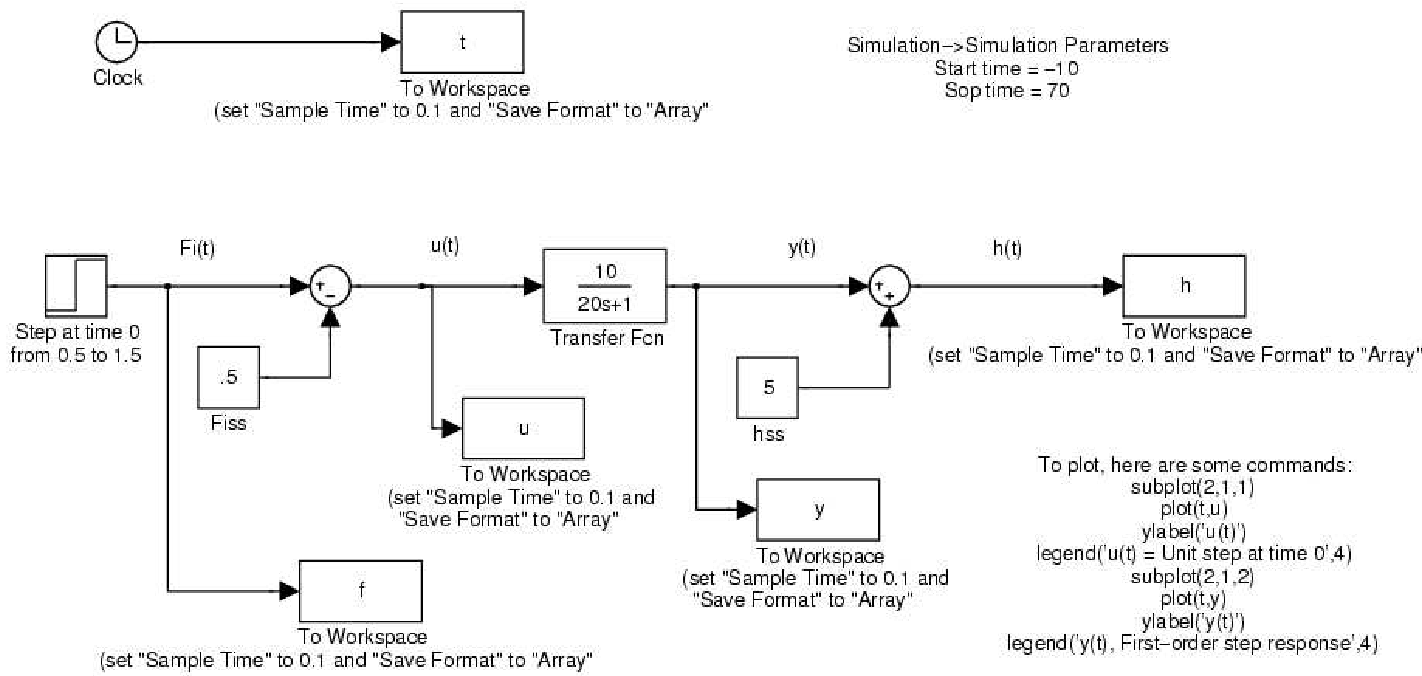
$$y(t|_{t=20}) = 10\left(1 - e^{-1}\right)$$

$$y(t|_{t=20}) = 10\,(1 - 0.3678)$$

$$y(t|_{t=20}) = 10\,(0.6321)$$

$$y(|_{t=20}) = 6.32$$

So at time $t = \tau$ the response is 6.32, or 63% of the final value of 10.
This can also be simulated in Simulink:



After running the simulation, the results will be put in vectors in the Matlab workspace. These vectors are named (in this example) t, y, u, h, and f. Note that the step occurs at time $t = 0$, so you should start the simulation at time $t = -10$. Also note that the "To Workspace" blocks must have the "Save Format" set to "Array".

The following plotting command will let you plot $u(t)$ and $y(t)$ on the same figure:

subplot(2,1,1)
plot(t,u)
ylabel('u(t)')
legend('u(t) = Unit step at time 0',4)
subplot(2,1,2) plot(t,y)
ylabel('y(t)')
legend('y(t), First-order step response',4)