Chapter 4

Feedback Linearizing Control

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4.1 Introduction

In this chapter, the basic theory of feedback linearization is presented and issues of particular relevance to process control applications are discussed. Two fundamental nonlinear controller design techniques — input-output linearization and state-space linearization — are discussed in detail. The theory also is presented for linear systems to facilitate understanding of the nonlinear results. Extensions are presented for disturbances and multivariable processes. Advanced topics such as dynamic feedback linearization, time delay compensation, constraint handling, robustness, and sampled-data systems are also discussed.

A survey of process control strategies and applications shows that: (1) a variety of nonlinear controller design techniques are based on input-output linearization; (2) few experimental studies of these techniques have been presented; and (3) many important problems remain unsolved. To illustrate design and implementation issues, feedback linearizing controllers are developed for three representative processes: a continuous stirred tank reactor, a continuous fermentor, and a pH neutralization system. This chapter differs from existing reviews [69, 83, 90, 95, 99, 100, 126, 134] of feedback linearization by providing a balanced discussion of theoretical and practical issues of interest to process control engineers.

Nonlinear Process Model

As discussed in previous chapters, there are several types of finite-dimensional, nonlinear process models. In this chapter, we will focus on continuous-time, state-space models of the form,

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{4.1}$$

where: x is an n-dimensional vector of state variables; u is an m-dimensional vector of manipulated input variables; y is an m-dimensional vector of controlled output variables; $f(x)$ is an n-dimensional vector of nonlinear functions; $g(x)$ is an $(n\text{x}m)$-dimensional matrix of nonlinear functions; and $h(x)$ is an m-dimensional vector of nonlinear functions. The single-input, single-output (SISO) case where $m = 1$ will be emphasized to facilitate understanding of the basic concepts. The model (4.1) will be modified as necessary to describe more complex nonlinear processes, such as those with measured disturbances or time delays.

Feedback Linearization vs. Jacobian Linearization

Consider the Jacobian linearization of the nonlinear model (4.1) about an equilibrium point (u_0, x_0, y_0):

$$\begin{aligned} \dot{x} &= \left[\frac{\partial f(x_0)}{\partial x} + \frac{\partial g(x_0)}{\partial x} u_0\right](x - x_0) + g(x_0)(u - u_0) \\ y - y_0 &= \frac{\partial h(x_0)}{\partial x}(x - x_0) \end{aligned} \tag{4.2}$$

Using deviation variables, the Jacobian model can be written as a linear state-space system,

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{4.3}$$

with obvious definitions for the matrices A, B, and C. It is important to note that the Jacobian model is an exact representation of the nonlinear model only at the point (x_0, u_0). As a result, a control strategy based on a linearized model may yield unsatisfactoy performance and robustness at other operating points.

In this chapter, we present a class of nonlinear control techniques that can produce a linear model that is an *exact* representation of the original nonlinear model over a large set of operating conditions. The general approach — typically called *feedback linearization* — is based on two operations: (1) nonlinear change of coordinates; and (2) nonlinear state feedback. We focus on *local* feedback linearization (*i.e.* the coordinate transformation and control law may be only locally defined) to avoid complications associated with the global problem.

After feedback linearization, the input-output model is linear,

$$\begin{aligned} \dot{\xi} &= A\xi + Bv \\ w &= C\xi \end{aligned} \tag{4.4}$$

where: ξ is an r-dimensional vector of transformed state variables; v is an m-dimensional vector of transformed input variables; w is an m-dimensional vector of transformed output variables; and the matrices A, B, and C have a very simple canonical structure. If $r < n$, an additional n–r state variables must be introduced to complete the coordinate transformation. The integer r is called the *relative degree* and is a fundamental characteristic of a nonlinear system.

Feedback Linearization Approaches

Most feedback linearization approaches are based on *input-output linearization* or *state-space linearization*. In the input-output linearization approach, the objective is to linearize the map between the transformed inputs (v) and the *actual* outputs (y). A linear controller is then designed for the linearized input-output model, which can be represented by (4.4) with $r \leq n$ and $w = y$. However, there is an (n–r)-dimensional subsystem that typically is not linearized,

$$\dot{\eta} = q(\eta, \xi) \tag{4.5}$$

where η is an (n–r)-dimensional vector of transformed state variables and q is a (n–r)-dimensional vector of nonlinear functions. Input-output linearization techniques are restricted to processes in which these so-called *zero dynamics* are stable.

In the state-space linearization approach, the goal is to linearize the map between the transformed inputs and the *entire* vector of transformed state variables. This objective is achieved by deriving *artificial* outputs (w) that yield a feedback linearized model with state dimension $r = n$. A linear controller is then synthesized for the linear input-state model. However, this approach may fail to simplify the controller design task because the map between the transformed inputs and the original outputs (y) generally is nonlinear. As a result, input-output linearization is preferable to state-space linearization for most process control applications. For some processes, it is possible to simultaneously linearize the input-state and input-output maps because the original outputs yield a linear model with dimension $r = n$.

Static and Dynamic State Feedback

Feedback linearization produces a linear model by the use of nonlinear coordinate transformations and nonlinear state feedback. Coordinate transformations are described in Chapter 3; nonlinear state feedback is discussed below. In some applications, the control objectives can be achieved with a nonlinear static state feedback control law of the form,

$$u = \alpha(x) + \beta(x)v \tag{4.6}$$

where α is an m-dimensional vector of nonlinear functions and β is an m x m matrix of nonlinear functions. For some processes, it is not possible to satisfy the control objective with a static controller and a dynamic state feedback control law must be employed,

$$\begin{aligned} \dot{\zeta} &= \gamma(x,\zeta) + \delta(x,\zeta)v \\ u &= \alpha(x,\zeta) + \beta(x,\zeta)v \end{aligned} \tag{4.7}$$

where: ζ is a q-dimensional vector of controller state variables; γ is a q-dimensional vector of nonlinear functions; and δ is a q x m matrix of nonlinear functions. Specific forms for the nonlinear controller functions $(\alpha, \beta, \gamma, \delta)$ will be presented thoughout the chapter.

4.2 Input-Output Linearization

4.2.1 Linear System

An input-output linearizing controller is designed for the SISO version of the linear system (4.3). This exercise illustrates the basic controller design procedure employed in the nonlinear case. Recall from

Chapter 3 that the system can be transformed into the following normal form via a linear change of coordinates $[\xi^T, \eta^T]^T = Tx$,

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_r &= R\xi + S\eta + ku \\
\dot{\eta} &= P\xi + Q\eta \\
y &= \xi_1
\end{aligned} \tag{4.8}$$

where r is the *relative degree*. The static state feedback control law,

$$u = \frac{v - R\xi - S\eta}{k} \tag{4.9}$$

changes the r-th equation in (4.8) to: $\dot{\xi}_r = v$. The transformed input v is designed to stabilize the ξ subsystem,

$$v = -\alpha_r \xi_r - \alpha_{r-1}\xi_{r-1} - \cdots - \alpha_1 \xi_1 \tag{4.10}$$

In the original coordinates, the complete control law has the form,

$$u = \frac{-CA^r x - \alpha_r CA^{r-1}x - \cdots - \alpha_1 Cx}{CA^{r-1}b} \tag{4.11}$$

The proposed control law yields the following characteristic equation for the ξ subsystem:

$$s^r + \alpha_r s^{r-1} + \cdots + \alpha_2 s + \alpha_1 = 0 \tag{4.12}$$

Nominal stability of the ξ subsystem, and therefore boundedness of the output, is ensured if the controller tuning parameters α_i are chosen such that (4.12) is a Hurwitz polynomial. As discussed in Chapter 3, the closed-loop system is internally stable (*i.e.* the η variables are bounded) if and only if the eigenvalues of the matrix Q are in the open left-half plane. Hence, the proposed controller design technique is restricted to linear systems which are *minimum phase*.

4.2.2 Controller Design

In this section, an input-output linearizing controller is designed for the SISO version of the nonlinear system (4.1). Extensions of the basic controller design procedure for disturbances and multivariable processes are discussed in subsequent sections.

Illustrative Example

In order to illustrate the basic concepts, we first consider the following two-dimensional nonlinear system:

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2) + g_1(x_1, x_2)u \\
\dot{x}_2 &= f_2(x_1, x_2) \\
y &= x_1
\end{aligned} \tag{4.13}$$

This model form can describe, for example, an irreversible reaction occuring in a constant volume, stirred tank reactor where x_1 is the reactor temperature, x_2 is the reactor concentration, and u is the coolant temperature [162].

If the nonlinear function g_1 is non-zero in the operating region of interest, the static state feedback control law,

$$u = \frac{v - f_1(x_1, x_2)}{g_1(x_1, x_2)} \tag{4.14}$$

changes the first equation of (4.13) to: $\dot{x}_1 = v$. Thus, the control law *exactly* linearizes the map between the transformed input v and the output y. Consequently, a *linear* controller can be designed to satisfy control objectives such as setpoint tracking. It is important to note that the x_2 dynamics remain nonlinear. As discussed in the next section, asymptotic stability of the these *zero dynamics* is a necessary condition for nominal closed-loop stability.

General Design Procedure

We now consider the design of an input-output linearizing controller for the n-th order nonlinear system (4.1). This problem was originally posed and solved by Isidori and Krener [85, 106]. Recall from Chapter 3 that the system (4.1) can be transformed into normal form via a diffeomorphism $[\xi^T, \eta^T]^T = \Phi(x)$ if the relative degree r is well defined. The ξ coordinates are defined as,

$$\xi_k = \Phi_k(x) = L_f^{k-1} h(x), \qquad 1 \le k \le r \tag{4.15}$$

and $\eta_k = \Phi_{r+k}(x),\ 1 \le k \le n - r$, where $L_g \Phi_k(x) = 0$. The normal form can be written as:

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_r &= b(\xi, \eta) + a(\xi, \eta) u \\ \dot{\eta} &= q(\xi, \eta) \\ y &= \xi_1 \end{aligned} \tag{4.16}$$

The static state feedback control law,

$$u = \frac{v - b(\xi, \eta)}{a(\xi, \eta)} \tag{4.17}$$

changes the r-th equation of (4.16) to: $\dot{\xi}_r = v$. As a result, the map between the transformed input v and the output y is *exactly* linear. Thus, a linear state feedback controller can be synthesized to stabilize the ξ subsystem. For instance, the pole placement design (4.10) yields the characteristic polynomial (4.12) for the linear subsystem. When expressed in the original coordinates, the two control laws have the following form:

$$u = \frac{v - L_f^r h(x)}{L_g L_f^{r-1} h(x)} \tag{4.18}$$

$$v = -\alpha_r L_f^{r-1} h(x) - \alpha_{r-1} L_f^{r-2} h(x) - \cdots - \alpha_1 h(x) \tag{4.19}$$

Hence, the complete static state feedback control law can be written as:

$$u = \frac{-L_f^r h(x) - \alpha_r L_f^{r-1} h(x) - \cdots - \alpha_1 h(x)}{L_g L_f^{r-1} h(x)} \tag{4.20}$$

Integral Control

In most process control applications, the objective is to maintain the output at a non-zero setpoint despite unmeasured disturbances and plant/model mismatch. Consequently, the nonlinear controller (4.20) should contain an integral term that penalizes deviations between the output (y) and its setpoint (y_{sp}). The modified input-output linearizing controller can be written as,

$$u = \frac{-L_f^r h(x) - \alpha_r L_f^{r-1} h(x) - \cdots + \alpha_1 \left[y_{sp} - h(x)\right] + \alpha_0 \int_0^t \left[y_{sp} - h(x)\right] d\tau}{L_g L_f^{r-1} h(x)} \tag{4.21}$$

where α_0 is an additional controller tuning parameter associated with the integral term. The integral control law (4.21) yields the following characteristic equation for the ξ subsystem:

$$s^{r+1} + \alpha_r s^r + \cdots + \alpha_1 s + \alpha_0 = 0 \tag{4.22}$$

By choosing the controller parameters α_i in terms of a single tuning parameter ϵ, the following closed-loop transfer function is obtained for setpoint changes if $y(0) = y_{sp}(0)$ [68]:

$$\frac{y(s)}{y_{sp}(s)} = \frac{\frac{r+1}{\epsilon^r} s + 1}{(\epsilon s + 1)^{r+1}} \tag{4.23}$$

4.2.3 Nominal Stability

In this section, nominal stability of the closed-loop system resulting from input-output linearization is discussed. For simplicity, we focus on asymptotic stabilization rather than the more difficult problem of asymptotic setpoint tracking [83]. We will assume, without loss of generality, that $x_0 = 0$ is an equilibrium point. The objective is to find conditions which ensure that the origin is a locally or globally asymptotically stable equilbrium point of the input-output linearized system.

Consider the closed-loop system comprised of the nonlinear system (4.1) and the input-output linearizing controller (4.20). The closed-loop system has the following representation in the transformed coordinates,

$$\begin{aligned} \dot{\xi} &= A\xi \\ \dot{\eta} &= q(\xi, \eta) \\ y &= \xi_1 \end{aligned} \tag{4.24}$$

If the controller tuning parameters α_i are chosen such that the characteristic polynomial (4.12) is Hurwitz, then the linear state variables ξ converge exponentially to the origin for any initial state $\xi(0)$ for which the control law (4.17) remains well defined (*i.e.* $a \neq 0$).

For the moment, assume the nonlinear state variables η converge asymptotically to the origin. Recall that the original state variables are related to the transformed state variables as $[\xi^T, \eta^T]^T = \Phi(x)$. Further assume that the diffeomorphism Φ is well defined in the region of interest. In this case, Φ is smooth and invertible, and can be chosen such that $\Phi(0) = 0$. As a result, asymptotic convergence of ξ and η to the origin implies asymptotic convergence of x to the origin. Thus, the origin is a locally asymptotically stable equilibrium point if: (1) the η state variables are locally asymptotically stable; and (2) the diffeomorphism and input-output linearizing control law are locally defined. The origin is a globally asymptotically stable equilibrium point if these conditions hold globally. Hence, the stabilization problem is effectively reduced to finding conditions which guarantee that the η state variables converge asymptotically to the origin.

Local Stability

We first present a necessary and sufficient condition for the input-output linearized system to be locally asymptotically stable. Because the ξ state variables converge to zero, the second equation of (4.24) becomes,

$$\dot{\eta} = q(0, \eta) \tag{4.25}$$

in the limit as $t \to \infty$. Equation (4.25) is known as the *zero dynamics* and is the nonlinear analog of linear system zeros. Nonlinear systems with asymptotically stable zero dynamics are said to be "minimum phase." Local asymptotic stability of the zero dynamics is clearly a necessary condition for the feedback linearized system (4.24) to be locally asymptotically stable. It has been shown that this condition also is sufficient [27, 123].

Global Stability

It is tempting to conjecture that global asymptotic stability of the zero dynamics is a sufficient condition for the feedback linearized system (4.24) to be globally asymptotically stable. The argument for this proposition proceeds as follows. The ξ state variables can be forced to zero arbitrarily fast by approriate selection of the controller tuning parameters α_i. Once the ξ variables converge to zero, the closed-loop trajectories are described by the zero dynamics (4.25). Because the zero dynamics are globally asymptotically stable by assumption, the η state variables converge to zero and the closed-loop system is globally asymptotically stable.

In fact, this argument is correct if the relative degree $r = 1$ [26, 27]. However, the argument does not hold in general if $r \geq 2$ due to the so-called "peaking phenomenon" [159, 160]. A high gain linear feedback can cause the linear state variables ξ to become very large before they decay to zero. These "peaking" variables act as destabilizing inputs to the zero dynamics. As a result, considerably more restrictive sufficient conditions are required to ensure that the system (4.24) is globally asymptotically stable if $r \geq 2$. Peaking cannot occur — and the closed-loop system therefore is globally asymptotically stable — if either of the following conditions are satisfied:

1. The only linear state variable entering the zero dynamics is the output $y = \xi_1$ [118, 148].

2. The zero dynamic satisfy a Lipschitz growth condition [27, 160].

Although neither condition may hold in practice, the authors have not encountered any process models with stable zero dynamics that cannot be asymptotically stabilized via input-output linearization.

4.2.4 Disturbance Decoupling

The input-output linearization technique presented above does not explicitly address process disturbances. In this section, conditions under which the output can be completely *decoupled* from disturbance variables are presented. This problem was originally posed and solved by several investigators using differential geometric analysis tools [37, 76, 86]. For simplicity, we employ an alternative formulation based on relative degrees [40]. Consider an SISO nonlinear system with a single disturbance d,

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + p(x)d \\ y &= h(x) \end{aligned} \tag{4.26}$$

where $p(x)$ is an n-dimensional vector of nonlinear functions. The controller design technique presented below is easily extended to processes with multiple disturbances [40].

The *disturbance decoupling problem* is to find (if possible) a diffeomorphism and a nonlinear static state feedback control law such that: (1) the map between the transformed input and the output is linear; and (2) the output is completely unaffected by the disturbance. It is useful to define a relative degree for the disturbance that is analogous to the relative degree r associated with the manipulated input u. The disturbance d is said to have relative degree ρ at the point x_0 if:

1. $L_p L_f^k h(x) = 0$ for all x in a neighborhood of x_0 and all $k < \rho - 1$.
2. $L_p L_f^{\rho-1} h(x_0) \neq 0$.

If $\rho < r$, the disturbance affects the output more directly than does the manipulated input and disturbance decoupling cannot be achieved with a static state feedback control law. Consequently, a necessary condition for the solution of the disturbance decoupling problem is that $\rho \geq r$. Under this condition, the diffeomorphism in Section 5.2.2 transforms (4.26) into the normal form,

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_r &= b(\xi,\eta) + a(\xi,\eta)u + s(\xi,\eta)d \\ \dot{\eta} &= q(\xi,\eta) + t(\xi,\eta)d \\ y &= \xi_1 \end{aligned} \tag{4.27}$$

where: $s(\xi,\eta) = L_p L_f^{\rho-1} h\left[\Phi^{-1}(\xi,\eta)\right]$ and $t_i(\xi,\eta) = L_p \Phi_{r+k}\left[\Phi^{-1}(\xi,\eta)\right]$, $1 \leq k \leq n-r$.

Assume that an on-line measurement of the disturbance is *not* available. If $\rho > r$, the function $s = 0$ and the linearizing control law (4.17) completely decouples the output from the disturbance. The transformed input v can be designed as usual. The requirement that $\rho > r$ is often called the *disturbance matching condition*. Because the disturbance acts as an input to the zero dynamics, stability analysis is more difficult than in the disturbance-free case. Sufficient conditions for local asymptotic stabilization have been presented [25, 164].

Note that the control law (4.17) does not decouple the output from the disturbance if $\rho = r$. It is possible to achieve disturbance decoupling in this case if the disturbance d is measured and a feedforward/feedback control law is employed. The so-called *disturbance decoupling problem with measurement* is solved by the following nonlinear control law:

$$u = \frac{v - b(\xi,\eta) - s(\xi,\eta)d}{a(\xi,\eta)} \tag{4.28}$$

This feedforward/feedback control law changes the r-th equation in (4.27) to: $\dot{\xi}_r = v$. As a result, the input-output map is linear and the output is decoupled from the disturbance. When expressed in the original coordinates, the decoupling control law has the form:

$$u = \frac{v - L_f^r h(x) - L_p L_f^{r-1} h(x) d}{L_g L_f^{r-1} h(x)} \tag{4.29}$$

The transformed input v is designed as before. A more general solution to the disturbance decoupling problem with measurement has been proposed for the case $\rho < r$ [40]. However, the resulting control law contains derivatives of the disturbance up to order $r - \rho$. Because process measurements usually are corrupted with high frequency noise, this approach is difficult to successfully implement in practice.

4.2.5 Input-Output Decoupling

The input-output linearization approach is extended to multiple-input, multiple-output (MIMO) processes. This extension is often called *input-output decoupling* because the input-output response is both linearized and decoupled. More precisely, the input-output decoupling problem is to find (if possible) a diffeomorphism and a state feedback control law such that: (1) the map between the transformed inputs v and controlled outputs y is linear; and (2) the i-th output y_i is decoupled from all inputs v_j for $i \neq j$. In this section, we will only consider static state feedback control laws [53, 86, 133, 138]; input-output decoupling based on dynamic state feedback is discussed elsewhere [44, 132].

The Decoupling Matrix

It is useful to represent the MIMO nonlinear system (4.1) as,

$$\begin{aligned} \dot{x} &= f(x) + \sum_{j=1}^{m} g_j(x) u_j \\ y_i &= h_i(x) \quad i = 1, 2, \ldots, m \end{aligned} \tag{4.30}$$

The nonlinear system is said to have *vector relative degree* $\{r_1, r_2, \ldots, r_m\}$ at the point x_0 if:

1. $L_{g_j} L_f^k h_i(x) = 0$ for all $1 \leq i, j \leq m$, for all $k < r_i - 1$, and for all x in a neighborhood of x_0.
2. The m x m *decoupling matrix*

$$A(x) = \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \cdots & L_{g_m} L_f^{r_1-1} h_1(x) \\ \vdots & \ddots & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(x) & \cdots & L_{g_m} L_f^{r_m-1} h_m(x) \end{bmatrix} \tag{4.31}$$

   is nonsingular at the point x_0.

The integer r_i represents the smallest relative degree of the i-th output with respect to any of the m inputs. Nonsingularity of the decoupling matrix $A(x)$ may be viewed as the MIMO generalization of the condition that $L_g L_f^{r-1} h(x_0) \neq 0$. It is a necessary and sufficient condition for the solution of the input-output decoupling control problem with static state feedback. A constructive proof of sufficiency is provided below; necessity is proven elsewhere [83, 133].

Normal Form

We develop a MIMO generalization of the normal form introduced in Chapter 3. The sum of the individual relative degrees is defined to be: $r = r_1 + \cdots + r_m$. Because $A(x)$ is nonsingular, the diffeomorphism $[\xi^T, \eta^T] = \Phi(x)$ can be constructed by choosing the first r coordinates as,

$$\xi_k^i = \Phi_k^i(x) = L_f^{k-1} h_i(x) \tag{4.32}$$

where $1 \leq k \leq r_i$ and $1 \leq i \leq m$. It is always possible to find $n - r$ additional coordinates $\eta^T = [\Phi_{r+1}(x), \ldots, \Phi_n(x)]$ such that,

$$\Phi^T(x) = \left[\Phi_1^1(x), \ldots, \Phi_{r_1}^1(x), \ldots, \Phi_1^m(x), \ldots, \Phi_{r_m}^m(x), \Phi_{r+1}(x), \ldots, \Phi_n(x)\right] \tag{4.33}$$

is invertible at the point x_0 [83]. If the vector fields $g_1(x), g_2(x), \ldots, g_m(x)$ are involutive (see Section 3.2.6), it is possible to choose the additional coordinates such that $L_{g_j} \Phi_{r+i}(x) = 0$ for all $1 \leq i \leq n - r$ and for all $1 \leq j \leq m$.

In general, the involutivity condition is not satisfied and the normal form is,

$$\begin{aligned}
\dot{\xi}_1^i &= \xi_2^i \\
\dot{\xi}_2^i &= \xi_3^i \\
&\vdots \\
\dot{\xi}_{r_i}^i &= b_i(\xi, \eta) + \sum_{j=1}^{m} a_{ij}(\xi, \eta) u_j \\
\dot{\eta} &= q(\xi, \eta) + \sum_{j=1}^{m} p_j(\xi, \eta) u_j \\
y_i &= \xi_1^i
\end{aligned} \tag{4.34}$$

where $1 \leq i \leq m$. The functions $a_{ij}(\xi, \eta)$ are elements of the decoupling matrix A expressed in terms of the transformed coordinates; the functions b_i are defined as,

$$b_i(\xi, \eta) = L_f^{r_i} h_i \left[\Phi^{-1}(\xi, \eta)\right] \tag{4.35}$$

the vector q is defined as in the SISO case; and the vectors $p_j(\xi, \eta)$ are defined as,

$$p_j(\xi, \eta) = \begin{bmatrix} L_{g_j} \Phi_{r+1} \left[\Phi^{-1}(\xi, \eta)\right] \\ \vdots \\ L_{g_j} \Phi_n \left[\Phi^{-1}(\xi, \eta)\right] \end{bmatrix} \tag{4.36}$$

The r_i-th equations in the normal form can be collected and written as:

$$\begin{bmatrix} \dot{\xi}^1_{r_1} \\ \vdots \\ \dot{\xi}^m_{r_m} \end{bmatrix} = b(\xi, \eta) + A(\xi, \eta) u \tag{4.37}$$

Controller Design

Because the decoupling matrix is nonsingular by assumption, the static state feedback control law which achieves input-output decoupling can be derived directly from (4.37),

$$u = A^{-1}(\xi, \eta) \left[v - b(\xi, \eta) \right] \tag{4.38}$$

where v is an m-dimensional vector of transformed input variables. This control law — which can be interpreted as the MIMO generalization of the input-output linearizing control law (4.17) — changes (4.37) to:

$$\begin{bmatrix} \dot{\xi}^1_{r_1} \\ \vdots \\ \dot{\xi}^m_{r_m} \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \tag{4.39}$$

As a result, the input-output response is both *linear* and *decoupled*,

$$\begin{aligned} \dot{\xi}^i_1 &= \xi^i_2 \\ \dot{\xi}^i_2 &= \xi^i_3 \\ &\vdots \\ \dot{\xi}^i_{r_i} &= v_i \\ y_i &= \xi^i_1 \end{aligned} \tag{4.40}$$

where $1 \leq i \leq m$. Each input v_i can be chosen as in the SISO case:

$$v_i = -\alpha^i_{r_i} \xi^i_{r_i} - \alpha^i_{r_i - 1} \xi^i_{r_i - 1} - \cdots - \alpha^i_1 \xi^i_1 \tag{4.41}$$

When expressed in the original coordinates, the decoupling control law (4.38) has the form,

$$u = -A^{-1}(x) b(x) + A^{-1}(x) v \tag{4.42}$$

where $A(x)$ is the decoupling matrix (4.31) and the m-dimensional vector $b(x)$ has elements $L_f^{r_i} h_i(x)$.

Nominal Stability

The input-output decoupling control law generally provides only partial linearization of the closed-loop system. An expression for the remaining (n–r)-dimensional nonlinear subsystem can be derived from (4.34) and (4.38),

$$\dot{\eta} = q(\xi, \eta) - P(\xi, \eta) A^{-1}(\xi, \eta) b(\xi, \eta) + P(\xi, \eta) A^{-1}(\xi, \eta) v \tag{4.43}$$

where the definition of the matrix P follows directly from (4.34). If each input v_i is designed as (4.41), the nonlinear subsystem can be written as $\dot{\eta} = \tilde{q}(\xi, \eta)$. The zero dynamics are obtained by setting $\xi(t) = 0$ for all $t \geq 0$:

$$\dot{\eta} = \tilde{q}(0, \eta) \tag{4.44}$$

As in the SISO case, local asymptotic stability of the zero dynamics is a sufficient condition for local asymptotic stability of the closed-loop system [27]. By contrast, this condition is not necessary for MIMO systems [84]. Sufficient conditions for an input-output decoupled system to be globally asymptotically stable are presented elsewhere [27, 148, 160].

4.3 State-Space Linearization

4.3.1 Linear System

The state-space linearization approach is applied to the SISO linear system (4.3). Although feedback linearization obviously is unnecessary in the linear case, this exercise illustrates the basic concepts of the nonlinear design procedure. First, we attempt to determined an output function,

$$\hat{y} = \hat{C}x \tag{4.45}$$

such that the relative degree with respect to $\hat{y}$ is equal to the system order: $r = n$. The variable $\hat{y}$ is called an *artificial output* because it generally is different than the controlled output y in (4.3). The artificial output can be derived by solving the following set of n linear equations for the vector $\hat{C}$,

$$\hat{C} \begin{bmatrix} B & AB & \cdots & A^{n-2}B & A^{n-1}B \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \beta \end{bmatrix} \tag{4.46}$$

where β is a non-zero value. A solution exists if and only if the matrix $[B \ AB \ \cdots \ A^{n-1}B]$ is full rank; *i.e.* if and only if (A, B) are a controllable pair [28]. In this case, the solution is not unique because $\hat{C}A^{n-1}B$ can be any non-zero value.

We now derive the control law assuming that an artificial output has been determined. Consider the following change of coordinates:

$$\xi = Tx = \begin{bmatrix} \hat{C} \\ \hat{C}A \\ \vdots \\ \hat{C}A^{n-1} \end{bmatrix} x \tag{4.47}$$

It is easy to show that the linear system has the following normal form representation in the transformed coordinates,

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_n &= R\xi + ku \\ y &= \xi_1 \end{aligned} \tag{4.48}$$

where $R = \hat{C}A^nT^{-1}$ and $k = CA^{n-1}B$. The static state feedback control law,

$$u = \frac{v - R\xi}{k} \tag{4.49}$$

changes the last equation in (4.48) to: $\dot{\xi}_n = v$. The input v is designed to stabilize the transformed system:

$$v = -\alpha_n\xi_n - \alpha_{n-1}\xi_{n-1} - \cdots - \alpha_1\xi_1 \tag{4.50}$$

In the original coordinates, the controller has the following form,

$$u = \frac{-\hat{C}A^nx - \alpha_n\hat{C}A^{n-1}x - \cdots - \alpha_1\hat{C}x}{\hat{C}A^{n-1}B} \tag{4.51}$$

The transformed system has the following closed-loop characteristic equation:

$$s^n + \alpha_n s^{n-1} + \cdots + \alpha_2 s + \alpha_1 = 0 \tag{4.52}$$

Nominal stability of the transformed system — and therefore the original system — is guaranteed by choosing the controller tuning parameters α_i such that the characteristic polynomial is Hurwitz. recall that nominal stability for the input-output linearization approach is ensured only if the linear system is minimum phase. On the other hand, the state-space linearization approach has the following disadvantages: (1) the linear system must be controllable; and (2) it may be difficult to satisfy output tracking objectives since the relationship between the *actual* output and the transformed state variables is: $y = CT^{-1}\xi$. We will see that these distinctions also hold in the nonlinear case.

4.3.2 Controller Design

The *state-space linearization* problem is to find (if possible) a diffeomorphism and a static state feedback control law such that the map between the transformed input and *entire* vector of transformed state variable is linear. This problem was orginally posed by Korobov [93]; a complete solution for the single-input case was provided by Brockett [24]. An alternative solution that facilitates the construction of the linearizing transformations also has been developed [82, 157]. In this section, the solution of the state-space linearization problem for single-input systems is presented; the multivariable case is discussed elsewhere [81, 87, 88]. An extension of the controller design procedure for disturbances is discussed in a later section.

Illustrative Example

For simplicity, we first consider the two-dimensional nonlinear system (4.13). In this case, an artificial output which yields a relative degree $r = 2$ can be determined by inspection: $\hat{y} = \hat{h}(x) = x_2$. If the diffeomorphism $\xi = \Phi(x)$ is chosen as,

$$\begin{aligned} \xi_1 &= \hat{h}(x) = x_2 \\ \xi_2 &= L_f\hat{h}(x) = f_2(x_1, x_2) \end{aligned} \tag{4.53}$$

the system has the following normal form representation:

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= L_f^2 \hat{h}[\Phi^{-1}(\xi)] + L_g L_f \hat{h}[\Phi^{-1}(\xi)]u \\
\hat{y} &= \xi_1
\end{aligned}
\tag{4.54}
$$

Assuming $L_g L_f \hat{h}$ is non-zero in the region of interest, the static state feedback control law,

$$
u = \frac{v - L_f^2 \hat{h}[\Phi^{-1}(\xi)]}{L_g L_f \hat{h}[\Phi^{-1}(\xi)]}
\tag{4.55}
$$

yields: $\dot{\xi}_2 = v$. Thus, the map between the transformed input v and the transformed state vector ξ is linear. The input v can be used to design a linear controller for the feedback linearized system. It is important to note that the map between ξ and the actual output y generally is *nonlinear*.

General Design Procedure

We now consider the n-dimensional nonlinear system (4.1). For the moment, assume that an artificial output $\hat{y} = \hat{h}(x)$ that yields $r = n$ has been determined. The existence and construction of such an output are discussed below. Consider the coordinate transformation:

$$
\xi_k = \Phi_k(x) = L_f^{k-1} \hat{h}(x), \qquad 1 \le k \le n
\tag{4.56}
$$

In the new coordinates, the system has the following normal form representation:

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_n &= b(\xi) + a(\xi)u \\
\hat{y} &= \xi_1
\end{aligned}
\tag{4.57}
$$

where $a(\xi) = L_g L_f^{n-1} \hat{h}[\Phi^{-1}(\xi)]$ and $b(\xi) = L_f^n \hat{h}[\Phi^{-1}(\xi)]$. If the function $a \neq 0$ throughout the region of operation, the static state feedback control law,

$$
u = \frac{v - b(\xi)}{a(\xi)}
\tag{4.58}
$$

changes the n-th equation of the normal form to: $\dot{\xi}_n = v$. As a result, the map between the transformed input and each of the transformed state variables is linear. A linear state feedback controller can be synthesized for the state-space linearized system. For instance, the pole-placement design (4.50) yields the closed-loop characteristic polynomial (4.52). When expressed in the original coordinates, the complete control law is:

$$
u = \frac{-L_f^n \hat{h}(x) - \alpha_n L_f^{n-1} \hat{h}(x) - \cdots - \alpha_1 \hat{h}(x)}{L_g L_f^{n-1} \hat{h}(x)}
\tag{4.59}
$$

Artificial Output Determination

The controller design procedure presented above demonstrates that the existence of an artificial output that yields a maximal relative degree ($r = n$) is a sufficient condition for the solution of the state-space linearization problem. Although not shown here, this condition is also necessary [83]. We now present necessary and sufficient conditions for the existence of such an output. The proof of sufficiency provides a constructive procedure for determining a suitable output function.

Because the relative degree must equal the system order, it follows that the output function $\hat{h}(x)$ must satisfy:

$$\begin{aligned} &L_g L_f^{k-1} \hat{h}(x) = 0, \quad 1 \le k \le n-1 \\ &L_g L_f^{n-1} \hat{h}(x) \neq 0 \end{aligned} \tag{4.60}$$

The determination of $\hat{h}(x)$ from these conditions requires the solution of a set of partial differential equations that include derivatives up to order $n-1$. However, the conditions can be rewritten as a set of first-order partial differential equations using the Lie bracket operator introduced in Chapter 3 [83]:

$$\begin{aligned} &L_{ad_f^{k-1} g} \hat{h}(x) = 0, \quad 1 \le k \le n-1 \\ &L_{ad_f^{n-1} g} \hat{h}(x) \neq 0 \end{aligned} \tag{4.61}$$

Using the Frobenius theorem (see Chapter 3), it can be shown that a solution to these equations exists — and therefore the state-space linearization problem is solvable — if and only if the nonlinear systems is [157]:

1. Controllable – the matrix $\left[g(x) \ ad_f g(x) \ \cdots \ ad_f^{n-1} g(x) \right]$ has rank n.
2. Integrable – the vector fields $g(x), ad_f g(x), \ldots, ad_f^{n-2} g(x)$ are involutive.

Note that a set of vector fields $\{X_1(x), \ldots, X_p(x)\}$ is involutive if there exists scalar functions $\delta_{ijk}(x)$ such that:

$$ad_{X_i} X_j(x) = \sum_{k=1}^{p} \delta_{ijk}(x) X_k(x), \quad 1 \le i, j \le p, \ i \neq j \tag{4.62}$$

If the system is two-dimensional, the integrability condition is always satisfied [83] and only the controllability condition must be checked. Using the theory discussed in Chapter 3, it is easy to show that the two-dimensional system (4.13) is controllable if $L_g L_f \hat{h}(x) \neq 0$.

Practical Issues

A potential disadvantage of the state-space linearization approach is that the artificial output $\hat{y}$ generally is different than the controlled output y. As a result, y usually is a nonlinear function of the transformed state variables: $y = h[\Phi^{-1}(\xi)]$. In this case, it is difficult to design a state-space linearizing controller to satisfy output tracking objectives because desirable $\hat{y}$ behavior does not necessarily imply desirable y behavior. State-space linearization therefore is most appropriate for stabilization problems in which the controlled output is not specified *a priori*.

For instance, consider the incorporation of integral action into the state-space linearizing control law (4.59). By analogy to the input-output linearization approach, the following control law is proposed:

$$u = \frac{-L_f^n \hat{h}(x) - \alpha_n L_f^{r-1}\hat{h}(x) - \cdots + \alpha_1 \left[y_{sp} - \hat{h}(x)\right] + \alpha_0 \int_0^t \left[y_{sp} - \hat{h}(x)\right] d\tau}{L_g L_f^{n-1}\hat{h}(x)} \tag{4.63}$$

This control law guarantees that $\hat{y} \to y_{sp}$ as long as the closed-loop system is asymptotically stable. However, offset-free tracking of y will be achieved if and only if $h(x_0) = \hat{h}(x_0)$, where x_0 is the equilibrium point corresponding to y_{sp}. A linear state/output map can be ensured by reformulating the control objectives in terms of an output that is a linear combination of the ξ state variables. However, this approach is difficult or even impossible to employ in practice. In general, the state-space linearizing controller must provide simultaneous linearization of the input/state and state/output maps to have output tracking capabilities. Unfortunately, the necessary and sufficient conditions for complete linearization are considerably more restrictive than those for state-space linearization [35, 121].

Additional disadvantages of the state-space linearization approach are that the existence conditions may be difficult to verify and the partial differential equations (4.61) may be difficult to solve analytically. It is not possible to achieve complete linearization of the input/state map if either of the existence conditions is not satisfied. However, some degree of partial linearization usually can be achieved. For example, in the input-output linearization approach an r-dimensional subsystem is linearized. In some applications, it may be desirable to maximize the dimension of the linear subsystem. Maximal linearization can be achieved by constructing an output function which yields the highest possible relative degree. Necessary and sufficient conditions for the solution of this problem are available [106, 122].

4.3.3 Nominal Stability

Nominal stability analysis for the input-output linearization approach is complex due to the presence of a nonlinear subsystem (the zero dynamics) in the closed-loop system. By contrast, the state-space linearization approach yields complete linearization of the state equations. As a result, zero dynamics are not present and nominal stability analysis is much simpler. This is the single most important advantage of state-space linearization as compared to input-output linearization.

Consider the closed-loop system comprised of the nonlinear system (4.1) and the state-space linearizing controller (4.59). In the transformed coordinates, the closed-loop system can be written as,

$$\begin{aligned} \dot{\xi} &= A\xi \\ \hat{y} &= \xi_1 \end{aligned} \tag{4.64}$$

where the matrix A has the characteristic equation (4.52). If the control law is well defined (*i.e.* $L_g L_f^{n-1}\hat{h} \neq 0$), then the state variables ξ converge exponentially to the origin for any initial condition $\xi(0)$ as long as the controller tuning parameters α_i are chosen such that (4.52) is a Hurwitz polynomial. Because the diffeomorphism $\Phi(x)$ is invertible and satisfies $\Phi(0) = 0$, exponential convergence of ξ to the origin implies asymptotic convergence of the actual state variables x to the origin.

4.3.4 Disturbances

An extension of the state-space linearization approach for nonlinear systems with disturbances is presented below. This problem is discussed in a series of papers by Calvet and Arkun [29, 30, 32, 33]. For simplicity, we consider the SISO nonlinear system (4.26) with a single disturbance. However, similar results are available for processes with multiple disturbances [30].

As in the disturbance-free case, we attempt to find an artificial output $\hat{y} = \hat{h}(x)$ which yields a relative degree r equal to the system order n. Such an output exists if and only if the controllability and integrability conditions in Section 5.3.2 are satisfied. If an artificial output can be constructed, the relative degree ρ for the disturbance d can be defined as in the input-output linearization approach (Section 5.2.4). Note that the relative degrees will always satisfy $\rho \leq r$. The artificial output function $\hat{h}(x)$ is used to construct the diffeomorphism (4.56). For a general value of ρ, the nonlinear system (4.26) will have the following normal form,

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
&\vdots \\
\dot{\xi}_{\rho-1} &= \xi_\rho \\
\dot{\xi}_\rho &= \xi_{\rho+1} + \zeta_\rho(\xi) d \\
&\vdots \\
\dot{\xi}_{n-1} &= \xi_n + \zeta_{n-1}(\xi) d \\
\dot{\xi}_n &= b(\xi) + a(\xi) u + \zeta_n(\xi) d \\
\hat{y} &= \xi_1
\end{aligned} \tag{4.65}$$

where $a(\xi)$ and $b(\xi)$ are defined as in the disturbance-free case, and $\zeta_k(\xi) = L_p \Phi_k \left[\Phi^{-1}(\xi)\right]$.

Disturbance Decoupling

The normal form shows that the disturbance cannot be completely decoupled from the artificial output $\hat{y}$ with a state feedback control law. In fact, it is easy to prove that the disturbance decoupling problem (without measurement) cannot be solved for *any* output that is a linear combination of the transformed state variables. Most importantly, the state-space approach does not provide a systematic framework for deriving decoupling control laws for the *actual* output y. This is a significant disadvantage of state-space linearization as compared to input-output linearization, which yields disturbance decoupling if the relative degree of the actual output is less than ρ.

However, the disturbance decoupling problem with measurement can be solved if the matching condition $\rho = n$ holds. In this case, the following feedforward/state feedback control law is employed:

$$u = \frac{v - b(\xi) - \zeta_n(\xi) d}{a(\xi)} \tag{4.66}$$

This control law changes the final equation in the normal form (4.65) to: $\dot{\xi}_n = v$. Hence, the transformed system is completely linear and the input v can be designed as usual. When expressed in the original coordinates, the feedforward/feedback controller is:

$$u = \frac{v - L_f^n \hat{h}(x) - L_p L_f^{n-1} \hat{h}(x) d}{L_g L_f^{n-1} \hat{h}(x)} \tag{4.67}$$

However, this control law generally does not provide disturbance decoupling with respect to the actual output.

Stabilization of the Quasi-Linear System

In most process applications, the disturbance is unmeasured and/or the matching assumption $\rho = n$ does not hold. In this case, the state-space linearizing control law (4.58) yields a *quasi-linear* system,

$$\begin{aligned} \dot{\xi} &= A\xi + Bv + \zeta(\xi)d \\ \hat{y} &= C\xi \end{aligned} \tag{4.68}$$

where the definitions of A, B, C, and ζ follow directly from the feedback linearized version of the normal form (4.65). The objective is to design a *linear* controller that stabilizes the quasi-linear system (and therefore stabilizes the original nonlinear system). A controller synthesis procedure based on Lyapunov stability theory has been proposed [32, 33]. The technique is based on two rather restrictive assumptions: (1) the disturbance matching condition $\rho = n$ holds; and (2) the nonlinear vector function ζ satisfies a nonlinear growth condition. The first assumption can be relaxed, but this generally results in poor performance. The interested reader is referred to the original papers [32, 33] for further details.

4.4 Advanced Topics

In this section, several advanced topics on feedback linearizing controller design are discussed. We focus on the input-output linearization approach because it usually is more useful for process applications. Results are presented for the state-space linearization appproach as appropriate. For the sake of brevity, we omit several important topics such as approximate feedback linearization [63, 105], adaptive feedback linearization [89, 149], and non-minimum phase compensation [64, 161].

4.4.1 General Nonlinear Systems

We extend the input-output linearization approach to *general nonlinear systems* that are not necessarily affine in the manipulated input:

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned} \tag{4.69}$$

This problem is important because some processes are naturally described by non-affine models. Two alternative controller design strategies are presented [68, 163]. The first technique employs the non-affine nonlinear system (4.69) directly, while the second approach is based on an *extended system* that is control affine. Although not discussed here, similar results are available for the state-space linearization approach [134, 157].

Controller Design Based on the Original System

In this case, the input-output linearizing controller design is based directly on the non-affine system (4.69). We define the Lie derivative of the scalar function $h(x)$ with respect to the vector function $f(x, u)$ as:

$$L_f h(x, u) = \frac{\partial h(x)}{\partial x} f(x, u) \tag{4.70}$$

Higher-order Lie derivatives are defined as,

$$L_f^k h(x,u) = \frac{\partial L_f^{k-1} h(x,u)}{\partial x} f(x,u) \tag{4.71}$$

where: $L_f^0 h(x,u) = h(x)$. The system is said to have relative degree r at the point (x_0, u_0) if:

1. $\frac{\partial}{\partial u} L_f^k h(x,u) = 0$ for all x in a neighborhood of x_0, all u in a neighborhood of u_0, and all $k < r$.
2. $\frac{\partial}{\partial u} L_f^r h(x_0, u_0) \neq 0$.

It follows that the first r Lie derivatives do not depend explicitly on the input: $L_f^k h(x,u) = L_f^k h(x), \;\; 0 \leq k \leq r-1$.

The diffeormorphism $[\xi^T, \eta^T]^T = \Phi(x)$ that places the nonlinear system (4.69) in normal form is constructed as follows. The ξ variables are chosen as,

$$\xi_k = \Phi_k(x) = L_f^{k-1} h(x) \tag{4.72}$$

where $1 \leq k \leq r$. The η variables can be chosen as,

$$\eta_k = \Phi_{r+k}(x), \quad 1 \leq k \leq n-r \tag{4.73}$$

such that their time derivatives are independent of u: $L_f \Phi_{r+k}(x,u) = L_f \Phi_{r+k}(x)$. When expressed in the new coordinates, the nonlinear system has the following normal form representation,

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_r &= r(\xi, \eta, u) \\ \dot{\eta} &= q(\xi, \eta) \\ y &= \xi_1 \end{aligned} \tag{4.74}$$

where $r(\xi, \eta, u) = L_f^r h\left[\Phi^{-1}(\xi, \eta), u\right]$ and the $(n–r)$-dimensional vector q has components $q_k(\xi, \eta) = L_f \Phi_{r+k}\left[\Phi^{-1}(\xi,\right.$

The input-output linearizing control law is obtained by solving the following nonlinear algebraic equation for the input u:

$$r(\xi, \eta, u) = v \tag{4.75}$$

If this equation is solvable, the input-output response is linearized as the r-th equation in the normal form (4.74) becomes: $\dot{\xi}_r = v$. The transformed input v is designed as in the control affine case. When expressed in terms of the original state variables, the controller equation is:

$$L_f^r h(x,u) = v \tag{4.76}$$

In most cases, this equation will not have an analytical solution and therefore it must be solved on-line at each sampling point. For a solution to exist at the point (x_0, v_0), the following two conditions must be satisfied:

1. There exists a u_0 such that $L_f^r h(x_0, u_0) = v_0$.
2. $\frac{\partial}{\partial u} L_f^r h(x_0, u_0) \neq 0$.

The second condition holds since the system is assumed to have a well defined relative degree; the first condition may not hold. If both conditions are satisfied, the implicit function theorem [83] guarantees the existence of a locally defined static state feedback control law,

$$u = \Psi(x, v) \tag{4.77}$$

where the implicit function Ψ satisfies $L_f^r h\left[x, \Psi(x, v)\right] = v$.

Controller Design Based on an Affine System

In this case, the input-output linearizing controller design is based on an *extended system* that is control affine. A new manipulated input w is defined as $w = \dot{u}$, and u is viewed as a state variable. By defining the extended state vector as $\bar{x} = [x^T \; u]^T$, the nonlinear system (4.69) can be represented as,

$$\begin{aligned} \dot{\bar{x}} &= \bar{f}(\bar{x}) + \bar{g}(\bar{x}) w \\ y &= \bar{h}(\bar{x}) \end{aligned} \tag{4.78}$$

where:

$$\hat{f}(\bar{x}) = \begin{bmatrix} f(x, u) \\ 0 \end{bmatrix}, \quad g(\bar{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \bar{h}(\bar{x}) = h(x) \tag{4.79}$$

Due to the introduction of the integrator, the relative degree of the extended system is equal to $r + 1$, where r is the relative degree of the original non-affine system.

Because the extended system (4.78) is control affine, the input-output linearizing controller can be designed in the usual manner. The resulting control law has the following form when expressed in terms of the actual input u:

$$\dot{u} = \frac{v - L_{\bar{f}}^{r+1} \bar{h}(x, u)}{L_{\bar{g}} L_{\bar{f}}^r \bar{h}(x, u)} \tag{4.80}$$

A *dynamic* state feedback control law of the form (4.7) can be obtained by defining the controller state variable as $\zeta = u$:

$$\begin{aligned} \dot{\zeta} &= -\frac{L_{\bar{f}}^{r+1} \bar{h}(x, \zeta)}{L_{\bar{f}}^{r+1} \bar{h}(x, \zeta)} + \frac{1}{L_{\bar{f}}^{r+1} \bar{h}(x, \zeta)} v \\ u &= \zeta \end{aligned} \tag{4.81}$$

The transformed input v can be designed to place the closed-loop poles:

$$v = -\alpha_{r+1} L_{\bar{f}}^r \bar{h}(x, u) - \alpha_r L_{\bar{f}}^{r-1} \bar{h}(x) - \cdots - \alpha_1 \bar{h}(x) \tag{4.82}$$

4.4.2 Time Delay Compensation

Many nonlinear processes contain time delays due to transportation lags and measurement delays. In this section, an extension of the basic input-output linearizing design strategy for time delay systems is presented [74, 103]. A similar technique has been developed for the state-space linearization approach [80]. The SISO nonlinear system is assumed to have the form,

$$\begin{aligned} \dot{x} &= f(x) + g(x)u(t-\theta) \\ y &= h(x) \end{aligned} \tag{4.83}$$

where θ is the time delay associated with the manipulated input. Note that a time delay in the output can be handled simply by combining an input-output linearizing controller and a linear output predictor (*e.g.* Smith predictor) [74].

Motivation

We demonstrate that an input-output linearizing controller for the time delay system (4.83) requires future values of the process state variables. The system can be placed in the following normal form using the standard change of coordinates:

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_r &= b(\xi,\eta) + a(\xi,\eta)u(t-\theta) \\ \dot{\eta} &= q(\xi,\eta) \\ y &= \xi_1 \end{aligned} \tag{4.84}$$

The input-output linearizing control law is:

$$u(t-\theta) = \frac{v(t) - b(\xi,\eta)}{a(\xi,\eta)} \tag{4.85}$$

Hence, the current input $u(t)$ must be computed as:

$$u(t) = \frac{v(t+\theta) - b[\xi(t+\theta), \eta(t+\theta)]}{a[\xi(t+\theta), \eta(t+\theta)]} \tag{4.86}$$

When expressed in the original coordinates, the linearizing control law has the form:

$$u(t) = \frac{v(t+\theta) - L_f^r h[x(t+\theta)]}{L_g L_f^{r-1} h[x(t+\theta)]} \tag{4.87}$$

This control law cannot be implemented directly because $x(t+\theta)$ and $v(t+\theta)$ are not known at time t.

Controller Design

The following strategy is employed to make the linearizing control law (4.87) implementable. First, the future state vector $x(t+\theta)$ is replaced by the estimate $\hat{x}(t+\theta|t)$. The calculation of $\hat{x}(t+\theta|t)$ is discussed below. In addition, the future input $v(t+\theta)$ is replaced by the estimate $\hat{v}(t+\theta|t)$, which is computed as:

$$\hat{v}(t+\theta|t) = -\alpha_r L_f^{r-1}h[\hat{x}(t+\theta|t)] - \cdots - \alpha_1 h[\hat{x}(t+\theta|t)] + w(t) \tag{4.88}$$

Note that (4.88) is identical to (4.19) except that $\hat{x}(t+\theta|t)$ has been substituted for $x(t)$ and the signal $w(t)$ has been introduced. As discussed below, $w(t)$ is the output of a linear time delay compensator. The overall control law can be written as,

$$u(t) = \frac{w(t) - \sum_{k=1}^{r+1} \alpha_k L_f^{k-1} h[\hat{x}(t+\theta|t)]}{L_g L_f^{r-1} h[\hat{x}(t+\theta|t)]} \tag{4.89}$$

where $\alpha_{r+1} = 1$.

Under the assumption that $\hat{x}(t+\theta|t) = x(t+\theta)$ for all $t \geq 0$, it can be shown that the controller tuning parameters α_k can be chosen to yield the following closed-loop transfer function in the absence of plant/model mismatch [74]:

$$\frac{y(s)}{w(s)} = \frac{\epsilon^r e^{-\theta s}}{(\epsilon s + 1)^r} \tag{4.90}$$

A linear controller is designed for the transfer function (4.90) by considering w as the manipulated input. The following control law provides time delay compensation and integral action:

$$\frac{w(s)}{y_{sp}(s) - y(s)} = \frac{\epsilon^{-r}\,(\epsilon s + 1)^r}{(\epsilon s + 1)^r\, e^{-\theta s}} \tag{4.91}$$

This controller can be implemented as a linear Smith predictor [150]. By combining (4.90) and (4.91), it is easy to show that the overall transfer function for setpoint changes is,

$$\frac{y(s)}{y_{sp}(s)} = \frac{e^{-\theta s}}{(\epsilon s + 1)^r} \tag{4.92}$$

where ϵ is the controller tuning parameter.

Predictor Design

The remaining task is to compute the one-time-delay-ahead estimate of the process state vector $\hat{x}(t+\theta|t)$. The following prediction formula has been proposed [74],

$$\begin{aligned} \hat{x}(t+\theta|t) \quad &= \quad x(t) + \int_t^{t+\theta} \left[f[\hat{x}(\tau|t)] + g[\hat{x}(\tau|t)]u(\tau-\theta) \right] d\tau + \\ &\quad\quad \lambda \left[x(t) - \hat{x}(t|t-T_d) \right] \end{aligned} \tag{4.93}$$

where λ and T_d are tuning parameters and the predictor is initialized with the plant state $\hat{x}(t|t) = x(t)$. The filter parameter λ accounts for modeling errors; it should be maintained in the range $0 \leq \lambda < \frac{\theta}{T_d}$. It is chosen near the upper limit if there is little plant/model mismatch and near zero if significant modeling errors are present. The term $\hat{x}(t|t-T_d)$ in the prediction (4.93) is calculated as,

$$\hat{x}(t|t-T_d) = x(t-T_d) + \int_{t-T_d}^{t} [f[\hat{x}(\tau|t-T_d)] + g[\hat{x}(\tau|t-T_d)]u(\tau-\theta)]\, d\tau \tag{4.94}$$

where $\hat{x}(t-T_d|t-T_d) = x(t-T_d)$. If the model is perfect and the predictor is initialized as $\hat{x}(0) = x(\theta)$, it can be shown that perfect state predictions are obtained when $\lambda = 1$ [74].

4.4.3 Constrained Nonlinear Systems

Many processes have constraints on input and output variables. In this section, we present an input-output linearization strategy for constrained nonlinear systems [67]. The basic idea is to map the original constraints into constraints on the feedback linearized system. The resulting linear system is controlled with a linear model predictive controller with explicit constraint handling capability. For the sake of brevity, we do not discuss alternative constraint compensation schemes based on feedback linearization [30, 92, 131, 152].

Constraint Mapping

The controller design is based on the SISO nonlinear model (4.1) with the following constraints:

$$u_{min} \leq u \leq u_{max}, \quad \Delta u_{min} \leq \Delta u \leq \Delta u_{max}, \quad y_{min} \leq y \leq y_{max} \tag{4.95}$$

The objective is to transform these constraints into constraints on the feedback linearized system. First, the linear subsystem is discretized to facilitate the linear model predictive controller design,

$$\begin{aligned} \xi(k+1) &= A_d\xi(k) + B_d v(k) \\ y(k) &= C\xi(k) \end{aligned} \tag{4.96}$$

where A_d is an r x r matrix, B_d is an r x 1 vector, and C is an 1 x r vector. The eigenvalues of A_d are on the unit circle, but the pair (A_d,B_d) is controllable.

The next step is to map the constraints on the original nonlinear system into constraints on the discretized linear system (4.96). The output constraints for the two systems are identical since the output is not transformed as part of the controller design. By contrast, the constraints on the actual input u must be mapped into constraints on the transformed input v. This transformation must be performed at each sampling instant because the mapping is state dependent (shown below). Moreover, the transformation must be performed for the entire control horizon N of the predictive controller. The constraint mapping is performed using the feedback linearizing control law (4.18) and the current measurement of the process state vector $x(k)$:

$$v(k) = L_f^r h[x(k)] + L_g L_f^{r-1} h[x(k)]u(k) = b[x(k)] + a[x(k)]u(k) \tag{4.97}$$

The constraints on the first input in the control horizon, $v(k|k)$, are calculated as,

$$\begin{aligned} v_{min}(k|k) &= \min_{u(k)} \quad b[x(k)] + a[x(k)]u(k) \\ v_{max}(k|k) &= \max_{u(k)} \quad b[x(k)] + a[x(k)]u(k) \end{aligned} \tag{4.98}$$

where:

$$\begin{aligned} u_{min} \leq \quad & u(k) \quad \leq u_{max} \\ \Delta u_{min} + u(k-1) \leq \quad & u(k) \quad \leq \Delta u_{max} + u(k-1) \end{aligned} \tag{4.99}$$

This optimization problem is trivial to solve since the objective function is affine in $u(k)$.

By contrast, computing the constraints for future inputs in the control horizon, $v(k+1|k), \ldots, v(k+N-1|k)$, is problematic. This difficulty occurs because future values of the input and state variables are needed to compute the future constraints, as indicated by the mapping (4.97). A simple way to overcome this problem is to extend the constraints calculated for $v(k|k)$ over the entire control horizon. An important property of this technique is that the *implemented* input,

$$u(k) = \frac{v(k|k) - L_f^r h[x(k)]}{L_g L_f^{r-1} h[x(k)]} \tag{4.100}$$

is guaranteed to satisfy the *actual* input constraints. On the other hand, the constraints calculated for the future inputs (which are not implemented) generally will not agree with the actual constraints. This is a serious disadvantage since incorrect future constraints may lead to implemented control moves that are overly conservative or aggressive. More accurate constraints can be obtained by utilizing the control moves calculated at the previous iteration of the predictive controller [67].

Linear Model Predictive Controller Design

The constrained linear model used for model predictive controller design has the form,

$$\begin{aligned} \xi(k+1) &= A_d \xi(k) + B_d v(k), \qquad v_{min}(k) \leq v(k) \leq v_{max}(k) \\ y(k) &= C\xi(k), \qquad y_{min} \leq y(k) \leq y_{max} \end{aligned} \tag{4.101}$$

It is important to note that the input constraints are *time varying*. The linear model is used to predict the effects of future control moves on future values of the output. To obtain improved predictions in the presence of plant/model mismatch, at each time step the linear model state is initialized with the current plant state:

$$\xi(k|k) = \begin{bmatrix} h[x(k)] \\ L_f h[x(k)] \\ \vdots \\ L_f^{r-1} h[x(k)] \end{bmatrix} \tag{4.102}$$

The matrix A_d is unstable because all its eigenvalues are located on the unit circle. Consequently, a model predictive controller design technique specifically developed for unstable linear system is employed [130]. At each time step, the manipulated input is generated by solving the following open-loop optimal control problem [130],

$$\begin{aligned} \min_{V(k)} \quad & s\left[v(k+N-1|k) - v_s\right]^2 + \sum_{j=0}^{N-1} \left[\xi(k+j|k) - \xi_s\right]^T Q \left[\xi(k+j|k) - \xi_s\right] + \\ & r\left[v(k+j|k) - v_s\right]^2 + s\left[v(k+j|k) - v(k+j-1|k)\right]^2 \end{aligned} \tag{4.103}$$

where: $\hat{\xi}(k+j|k)$ is the predicted value of $\xi(k+j)$; ξ_s and v_s are target values for ξ and v, respectively; $r > 0$ and $s \geq 0$ are scalar tuning parameters; and Q is a positive semidefinite tuning matrix. The decision vector $V(k)$ contains N values of the manipulated input: $V(k) = [v(k|k), v(k+1|k), \ldots, v(k+N-1|k)]^T$. Inputs beyond the control horizon are set equal to the target value: $v(k+j|k) = v_s,\ j \geq N$.

The targets ξ_s and v_s are calculated from the steady-state form of (4.101) under the condition that $y = y_{sp}$. The target values must lie within the feasible region defined by the input and output constraints for the optimal control problem to have a solution. In practice, a disturbance model is used to shift the targets in order to eliminate offset [67]. Because all the eigenvalues of A_d are located on the unit circle, a necessary condition for the optimization problem to have a solution is that the state variables are driven to their target values by the end of the control horizon: $\xi(k+N|k) = \xi_s$. Thus, the optimization problem must be solved subject to the following constraints:

$$\begin{aligned}
&v_{min}(k) \leq v(k+j|k) \leq v_{max}(k), \quad 0 \leq j \leq N-1 \\
&y_{min} \leq C\xi(k+j|k) \leq y_{max}, \quad j \geq 1 \\
&\xi(k+N|k) = \xi_s
\end{aligned} \tag{4.104}$$

If necessary, input and /or output constraints can be relaxed to ensure that the optimization problem is feasible [67, 145]. The optimal control problem (4.103) can be manipulated to yield a quadratic program that can be solved with standard software [130].

4.4.4 Robust Controller Design

We discuss robust controller design techniques based on the input-output and state-space linearization approaches. This is a very important topic because modeling errors usually preclude exact cancellation of nonlinear terms. The SISO nonlinear system is assumed to be described by the nominal nonlinear model (4.1) with state dependent perturbations:

$$\begin{aligned}
\dot{x} &= f(x) + \Delta f(x) + [g(x) + \Delta g(x)]\, u \\
y &= h(x)
\end{aligned} \tag{4.105}$$

Note that the perturbations $\Delta f(x)$ and $\Delta g(x)$ are *not* known. Available controller design techniques differ according to the *structure matching conditions* and *growth conditions* imposed on the perturbations. Less restrictive matching conditions generally require more restrictive bounds to be placed on the perturbations.

Input-Output Linearization Techniques

First we define relative degrees for the state dependent perturbations. The perturbation $\Delta f(x)$ is said to have relative degree γ_f at the point x_0 if:

1. $L_{\Delta f} L_f^k h(x) = 0$ for all x in a neighborhood of x_0 and all $k < \gamma_f - 1$.

2. $L_{\Delta f} L_f^{\gamma_f - 1} h(x_0) \neq 0$.

The relative degree γ_g for the pertubation $\Delta g(x)$ is defined analogously. For the moment, assume that the structure matching conditions $\gamma_f \geq r$ and $\gamma_g \geq r$ are satisfied. Using the standard change of coordinates, the uncertain system (4.105) has the following normal form representation,

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_r &= b(\xi,\eta) + \Delta b(\xi,\eta) + [a(\xi,\eta) + \Delta a(\xi,\eta)]\, u \\
\dot{\eta} &= q(\xi,\eta) + r(\xi,\eta) + s(\xi,\eta)u \\
y &= \xi_1
\end{aligned} \tag{4.106}
$$

where the functions a, b, and q are defined as usual and:

$$
\begin{aligned}
\Delta a(\xi,\eta) &= L_{\Delta g} L_f^{r-1} h \left[\Phi^{-1}(\xi,\eta)\right] \\
\Delta b(\xi,\eta) &= L_{\Delta f} L_f^{r-1} h \left[\Phi^{-1}(\xi,\eta)\right] \\
r_k(\xi,\eta) &= L_{\Delta f} \Phi_{r+k} \left[\Phi^{-1}(\xi,\eta)\right], \quad 1 \le k \le n-r \\
s_k(\xi,\eta) &= L_{\Delta g} \Phi_{r+k} \left[\Phi^{-1}(\xi,\eta)\right], \quad 1 \le k \le n-r
\end{aligned} \tag{4.107}
$$

Under the more restrictive matching conditions that $\gamma_f > r$ and $\gamma_g > r$, the perturbations do not appear in the r-th equation in the normal form. As a result, the standard input-output linearizing control law (4.20) yields a stable closed-loop system if the perturbed zero dynamics are asymptotically stable [119].

On the other hand, plant/model mismatch should be considered explicitly in the controller design if $\gamma_f \le r$ and/or $\gamma_g \le r$. Several Lyapunov-based design techniques have been proposed for this important case. Kravaris and Palanki [101] assume the existence of scalar functions $\Delta f^*(x)$ and $\Delta g^*(x)$ that satisfy the matching conditions:

$$
L_{\Delta f} L_f^{r-1}(x) = g(x)\Delta f^*(x), \qquad \Delta g(x) = g(x)\Delta g^*(x) \tag{4.108}
$$

These conditions are rather restrictive as they require that the peturbations be expressed in terms of the input vector $g(x)$. Behtash [22] considers similar, but slightly less restrictive, matching conditions. Arkun and Calvet employ the following matching conditions,

$$
\Delta f(x) = \Delta f_1(x) + \Delta f_2(x), \qquad \Delta g(x) = g(x)\Delta g^*(x) \tag{4.109}
$$

where $\Delta f_1(x)$ is a matched uncertainty and $\Delta f_2(x)$ is an unmatched uncertainty with relative degree $\gamma_{f_2} = r$. A stabilizing input-output linearizing controller is design by solving an algebraic Riccati equation. Liao *et al.* decompose both perturbations into matched and unmatched parts:

$$
\Delta f(x) = g(x)\Delta f_1(x) + \Delta f_2(x), \qquad \Delta g(x) = g(x)\Delta g_1(x) + \Delta g_2(x) \tag{4.110}
$$

Restrictive bounds must be placed on the unmatched uncertainties $\Delta f_2(x)$ and $\Delta g_2(x)$ to guarantee closed-loop stability.

State-Space Linearization Techniques

Su *et al.* [158] have shown that the state-space linearization approach possesses some degree of robustness to modeling errors. The analysis does not require matching conditions, but it is assumed that the plant and model are "close" in a topological sense. Controller design techniques that explicitly address plant/model mismatch usually employ: (1) matching conditions on both $\Delta f(x)$ and $\Delta g(x)$; or (2) no matching condition on $\Delta f(x)$, but $\Delta g(x) = 0$. The first group of robust controller design techniques are based on the following matching conditions:

$$\Delta f(x) = g(x)\Delta f^*(x), \qquad \Delta g(x) = g(x)\Delta g^*(x) \tag{4.111}$$

In this case, the standard change of coordinates places the nonlinear system (4.1) in the normal form,

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_n &= b(\xi,\eta) + \Delta b(\xi,\eta) + [a(\xi,\eta) + \Delta a(\xi,\eta)]\, u \\ \hat{y} &= \xi_1 \end{aligned} \tag{4.112}$$

where the nonlinear functions Δa and Δb are defined as in (4.107) with $r = n$. Spong [155] uses L_2 control theory to synthesize a stabilizing controller for the feedback linearized system, while Ha and Gilbert [59] employ a Lyapunov-based design strategy.

Several robust controller design techniques are based on the alternative matching condition $\Delta g(x) = 0$. In this case, the perturbation $\Delta f(x)$ is allowed to be arbitrary at the expense of assuming exact knowledge of the input vector $g(x)$. Spong and Vidyasagar [156] have proposed a design strategy in which the stable factorization approach is applied to the feedback linearized system. A Lyapunov design technique based on the solution of a algebraic Riccati equation has been developed by Calvet and Arkun [31]. Doyle and Morari [49] have proposed a robust controller design strategy based on conic sector bounds and approximate state-space linearization.

4.4.5 Discrete-Time and Sampled-Data Systems

Feedback linearizing controller design for discrete-time and sampled-data nonlinear systems is discussed below. In many ways, the results parallel the continuous-time results presented throughout this chapter. However, the operations used to construct the discrete-time controllers are quite different. In addition, discrete-time and sampled-data systems offer unique possibilities (*e.g.* deadbeat control) and complexities (*e.g.* sampling) not encountered in the continuous-time case. For simplicity, we focus on SISO models of the form:

$$\begin{aligned} x(k+1) &= F[x(k), u(k)] \\ y(k) &= h[x(k)] \end{aligned} \tag{4.113}$$

Because most processes are inherently continous, the function F is obtained by discretizing a continuous-time model or identified directly from process data. The discrete-time model is not assumed to be control affine because: (1) exact sampling generally does not yield an affine model; and (2) an affine model usually does not simplify the controller design task.

Discrete-Time Systems

For the moment, we neglect the effects of sampling and discuss the feedback linearization of inherently discrete-time systems. Sampled-data systems are considered below. For the sake of brevity, we focus on the input-output linearization problem [110, 127]. Similar results are available for disturbance decoupling [55, 134], input-output decoupling [56, 134], and state-space linearization [110, 134].

The *composition* of the scalar function $h(x)$ and the vector function $F(x)$ is defined as: $h \circ F(x) = h[F(x)]$. Higher order compositions are defined recursively: $h \circ F^i(x) = h \circ F^{i-1}[F(x)]$, where $h \circ F^0(x) = h(x)$. The composition operator plays the same role as does the Lie derivative in the continuous-time case. The discrete-time system (4.113) is said to have relative degree r at the point (x_0, u_0) if:

1. $\frac{\partial}{\partial u(k)} h \circ F^i[x(k), u(k)] = 0$ for all (x, u) in a neighborhood of (x_0, u_0) and all $i \leq r-1$.

2. $\frac{\partial}{\partial u(k)} h \circ F^r[x_0, u_0] \neq 0$.

By definition of the relative degree we can write,

$$h \circ F^i[x(k), u(k)] = h \circ F_0^i[x(k)], \quad 1 \leq k \leq r-1 \tag{4.114}$$

since these functions do not depend explicitly on the input $u(k)$.

If the relative degree is well defined, a diffeomorphism $[\xi^T(k), \eta^T(k)]^T = \Phi[x(k)]$ that places the system in normal form is constructed as follows. The ξ vaiables are chosen as,

$$\xi_i(k) = h \circ F_0^{i-1}[x(k)], \quad 1 \leq i \leq r \tag{4.115}$$

The remaining variables $\eta_i(k) = \Phi_{r+i}[x(k)]$, $1 \leq i \leq n-r$, can be chosen such that Φ is invertible and $\frac{\partial}{\partial u(k)} \Phi_i \circ F[x(k), u(k)] = 0$. As a result, the normal form is,

$$\begin{aligned} \xi_1(k+1) &= \xi_2(k) \\ \xi_2(k+1) &= \xi_3(k) \\ &\vdots \\ \xi_r(k+1) &= a[\xi(k), \eta(k), u(k)] \\ \eta(k+1) &= q[\xi(k), \eta(k)] \\ y(k) &= \xi_1(k) \end{aligned} \tag{4.116}$$

where:

$$\begin{aligned} a[\xi(k), \eta(k), u(k)] &= h \circ F^r[\Phi^{-1}(\xi(k), \eta(k)), u(k)] \\ q_i[\xi(k), \eta(k)] &= \Phi_{r+i} \circ F[\Phi^{-1}(\xi(k), \eta(k))], \quad 1 \leq i \leq n-r \end{aligned} \tag{4.117}$$

Note that the functions q_i do not depend on $u(k)$ by construction.

The input-output linearizing control law is obtained by solving the following nonlinear algebraic equation for $u(k)$,

$$a[\xi(k), \eta(k), u(k)] = v(k) \tag{4.118}$$

where $v(k)$ is the transformed input. This equation can be solved locally if the conditions of the implicit function theorem are satisfied (see Section 5.4.1). When expressed in terms of the original state variables, the linearizing control law is:

$$h \circ F^r[x(k), u(k)] = v(k) \tag{4.119}$$

If the transformed input is chosen as $v(k) = y_{sp}(k)$, output deadbeat control is obtained [120]: $y(k+r) = y_{sp}(k)$. Alternatively, the linear controller can be designed as,

$$v(z) = \frac{g_d(z)}{1 - z^{-r} g_d(z)} \left[y_{sp}(z) - y(z) \right] \tag{4.120}$$

where $g_d(z)$ is the desired closed-loop transfer function. A simple choice for g_d is,

$$g_d(z) = \frac{1-\alpha}{1-\alpha z^{-1}} \tag{4.121}$$

where the tuning parameter α represents the closed-loop pole. Note that the linear control law has integral action. For this design, the nominal closed-loop transfer function for setpoint changes is:

$$\frac{y(z)}{y_{sp}(z)} = z^{-r} \frac{1-\alpha}{1-\alpha z^{-1}} \tag{4.122}$$

As in the continuous-time case, closed-loop stability can be ensured only if the zero dynamics,

$$\eta(k+1) = q[0, \eta(k)] \tag{4.123}$$

are asymptotically stable.

Sampled-Data Systems

Feedback linearizing controller design becomes considerably more complex if the effects of sampling are considered. We briefly describe the potential difficulties and proposed solutions. Monaco and Normand-Cyrot [128] demonstrate that exact sampling always produces a discrete-time system of relative degree one. They also show that if the continuous-time system is minimum phase and has relative degree r, sufficiently fast sampling always yields a discrete-time system with: (1) stable zero dynamics if $r = 1$; and (2) unstable zero dynamics if $r \geq 3$. As a result, a discretized input-output linearizing controller is likely to provide satisfactory performance if $r = 1$, but it may produce unexpected results if $r \geq 2$. Glad [54] also considered sampling effects in developing output deadbeat controllers for sampled-data systems. The $r \geq 3$ case is handled by employing a multi-rate sampling strategy in which the control move is implemented each sampling period but the state is only measured every r sampling periods.

It has been shown that state-space linearizability of a continuous-time system can be destroyed by sampling [57]. Arapostathis *et al.* [14] demonstrate that sampling can impose severe restrictions — in addition to the standard controllability and involutivity conditions — on the structure of the continuous-time system. For two-dimensional systems, state-space linearizability of the sampled-data system implies that the continuous-time system can be completely linearized by coordinate transformations alone. To overcome these difficulties, several state-space linearization techniques based on multi-rate sampling have been proposed. Grizzle and Kokotovic [57] consider implementing the control move each sampling period while measuring the state every N sampling periods, where $N \geq 2$.

4.5 Process Control Strategies and Applications

A variety of nonlinear controller design strategies have been developed for process applications. Many of these techniques are based, either implicitly or explicitly, on exact linearization of the input-output response. In this section, these linearizing control strategies are reviewed and critically evaluated. Applications of feedback linearizing control techniques to process systems also are discussed. The applications are categorized in terms of the unit operations involved; a representative list of references is provided in each case. Additional applications are described in other review articles on feedback linearizing control [69, 90, 95, 100, 126].

4.5.1 Process Control Techniques

Techniques for Processes of Relative Degree One

Generic model control [111, 113], internal decoupling [16, 17] and reference system synthesis [19] are nonlinear controller design techniques developed specifically for process control applications. We outline the generic model control (GMC) design strategy and show that GMC is an input-output linearization technique for processes of relative degree one. Similar analyses of the internal decoupling and reference system synthesis techniques are omitted for the sake of brevity.

Consider the SISO nonlinear system (4.1) The rate-of-change of the output can be written as:

$$\dot{y} = L_f h(x) + L_g h(x) u \tag{4.124}$$

Assume that the desired rate-of-change is,

$$\dot{y}_d = k_1(y_{sp} - y) + k_2 \int_0^t (y_{sp} - y)\, d\tau \tag{4.125}$$

where k_1 and k_2 are controller tuning parameters. The goal is to determine a control law such that $\dot{y} = \dot{y}_d$ for all $t \geq 0$. If the relative degree $r = 1$, then the function $L_g h(x) \neq 0$ and the following state feedback control law achieves the control objective:

$$u = \frac{k_1(y_{sp} - y) + k_2 \int_0^t (y_{sp} - y)\, d\tau - L_f h(x)}{L_g h(x)} \tag{4.126}$$

The closed-loop system is input-output linearized,

$$\dot{y} = k_1(y_{sp} - y) + k_2 \int_0^t (y_{sp} - y)\, d\tau \tag{4.127}$$

and has the following transfer function for setpoint changes if $y(0) = y_{sp}(0)$:

$$\frac{y(s)}{y_{sp}(s)} = \frac{k_1 s + k_2}{s^2 + k_1 s + k_2} \tag{4.128}$$

The tuning parameters k_1 and k_2 are used to place the closed-loop poles. Note that the GMC control law (4.126) and the input-output linearizing control law (4.21) are *identical* when $r = 1$.

Globally Linearizing Control

Globally linearizing control (GLC) [96] is a controller design strategy developed for nonlinear process applications. We show that GLC is an input-output linearization technique for processes of arbitrary relative degree. Consider the SISO nonlinear system (4.1). By definition of the relative degree, the first r derviative of the output are:

$$\begin{aligned} y^{(k)} &= L_f^k h(x), \quad 1 \leq k \leq r-1 \\ y^{(r)} &= L_f^r h(x) + L_g L_f^{r-1} h(x) u \end{aligned} \tag{4.129}$$

The state feedback control law is chosen as,

$$u = \frac{\bar{v} - \beta_r L_f^r h(x) - \beta_{r-1} L_f^{r-1} h(x) - \cdots - \beta_0 h(x)}{\beta_r L_g L_f^{r-1} h(x)} \tag{4.130}$$

where $\bar{v}$ is a new input and β_k are controller tuning parameters.

It follows from (4.129) that the proposed control law yields a linearized input-output response:

$$\beta_r y^{(r)} + \beta_{r-1} y^{(r-1)} + \cdots + \beta_0 y = \bar{v} \tag{4.131}$$

A proportional-integral controller is designed for the feedback linearized system,

$$\bar{v} = k_c \left[(y_{sp} - y) + \frac{1}{\tau_I} \int_0^t (y_{sp} - y)\, d\tau \right] \tag{4.132}$$

where the gain k_c and integral time τ_I are additional controller tuning parameters. The complete GLC control law obtained by combining (4.130) and (4.132) yields the following closed-loop transfer function for setpoint changes if $y(0) = y_{sp}(0)$:

$$\frac{y(s)}{y_{sp}(s)} = \frac{k_c s + \frac{k_c}{\tau_I}}{\beta_r s^{r+1} + \beta_{r-1} s^r + \cdots + (\beta_0 + k_c) s + \frac{k_c}{\tau_I}} \tag{4.133}$$

The tuning parameters β_k, k_c, and τ_I are used to place the closed-loop poles.

The GLC control law is closely related to the input-output linearizing control law (4.21). In fact, the two control laws are *identical* if $\beta_r = 1$ and the tuning parameters of the input-output linearizing controller are chosen as:

$$\alpha_0 = \frac{k_c}{\tau_I}, \quad \alpha_1 = \beta_0 + k_c, \quad \alpha_k = \beta_{k-1},\ 2 \leq k \leq r \tag{4.134}$$

Although not discussed here, extensions of the GLC technique for measured disturbances [40], multivariable processes [102], and discrete-time systems [151] parallel those presented earlier for the input-output linearization approach.

Nonlinear Internal Model Control Techniques

Internal model control (IMC) is a powerful controller design strategy for linear process models [129]. Two distinctive characteristics of IMC are: (1) the controller design is based on the inverse of the process model; and (2) the error between the plant and model outputs is used as a feedback signal. Several nonlinear controller design techniques which include these two features have been proposed for process applications [51, 70, 137]. In this section, we discuss a particular nonlinear IMC technique [70] and provide comparisons with the input-output linearization approach.

As in the linear case, the nonlinear IMC control law is comprised of a model inverse controller C and a robustness filter F. For the moment, we assume $F = 1$ and design the controller C to optimize the nominal performance measure,

$$\min_C \parallel y_{sp}(t) - y(t) \parallel \tag{4.135}$$

where $\parallel \cdot \parallel$ represents a desired norm. The feedback signal to C is $e = y_{sp} - y + \tilde{y}$, where y is the plant output and $\tilde{y}$ is the model output. In the absence of modeling errors, $e = y_{sp}$ and the plant output can be written as $y = MCy_{sp}$, where M represents the process model. As a result, the controller design problem can be reformulated as:

$$\min_C \parallel (1 - MC)y_{sp}(t) \parallel \tag{4.136}$$

If the system is initially at rest and $y(0) = y_{sp}(0)$, the performance criterion is identically zero for any norm and setpoint trajectory when C is chosen to be the right-inverse of the model (see Chapter 3),

$$u = \frac{y_{sp}^{(r)} - L_f^r h(\tilde{x})}{L_g L_f^{r-1} h(\tilde{x})} \tag{4.137}$$

where $\tilde{x}$ are the model state variables.

The model inverse controller C is not suitable for implementation because: (1) "perfect" control requires unreasonably large control moves; (2) the controller is not "proper" in the sense that it requires derivatives of the setpoint; and (3) the perfect model assumption is not satisfied in practice. As a result, C is augmented with the following *nonlinear* filter F:

$$v^{(r)} = -\alpha_r L_f^{r-1} h(\tilde{x}) - \alpha_{r-1} L_f^{r-2} h(\tilde{x}) - \cdots - \alpha_1 h(\tilde{x}) + \alpha_1 e \tag{4.138}$$

The filter output $v^{(r)}$ replaces the signal $y_{sp}^{(r)}$ in the nonlinear control law (4.137). If the system is initially at rest and $y(0) = y_{sp}(0)$, the controller parameters α_k can be chosen to yield the nominal closed-loop transfer function:

$$\frac{y(s)}{y_{sp}(s)} = \frac{1}{(\epsilon s + 1)^r} \tag{4.139}$$

The controller tuning parameter ϵ can take values in the range $0 < \epsilon < \infty$. The nonlinear IMC technique possesses similar stability, perfect control, and zero offset properties as linear IMC [129]. In fact, the linear and nonlinear IMC design strategies yield identical controllers when applied to a stable, minimum-phase linear model.

The nonlinear IMC controller can be interpreted as a variant of the input-output linearizing controller (4.21). Despite the similarities, the nonlinear IMC technique offers several unique features including:

1. A control structure in which the difference between the plant and model outputs is used as a feedback signal for the nonlinear controller.

2. Implicit integral action that is a result of the IMC control structure.

3. An output feedback controller implementation because the nonlinear controller uses state variables generated by the model.

The third feature represents a significant advantage of the nonlinear IMC strategy as compared to the input-output linearization approach, which is based on full-state feedback. However, the nonlinear IMC technique employs the model as an open-loop observer and therefore it is restricted to open-loop stable processes.

4.5.2 Process Control Applications

Chemical Reactors

Feedback linearizing control strategies have been applied to a wide variety of chemical reactor models. The most commonly used model describes the irreversible, exothermic reaction $A \rightarrow B$, occurring in a constant volume, continuously stirred tank reactor [66]:

$$\begin{aligned}
\dot{C}_A &= \frac{q}{V}(C_{Af} - C_A) - k_0 \exp\left(-\frac{E}{RT}\right) C_A \\
\dot{T} &= \frac{q}{V}(T_f - T) + \frac{(-\Delta H)}{\rho C_p} k_0 \exp\left(-\frac{E}{RT}\right) C_A + \frac{UA}{V \rho C_p}(T_c - T)
\end{aligned} \tag{4.140}$$

The severe static and dynamic nonlinear behavior of this model are well documented [162]. The coolant temperature T_c or feed flowrate q is usually employed as the manipulated input, while either the reactor concentration C_A or temperature T is chosen as the controlled output. Other commonly used reactors models differ from (4.140) by including coolant jacket dynamics and/or by considering more complex reactions. A summary of applications of feedback linearization to chemical reactors is presented in Table 4.1. For each reference, the table contains the controller design approach employed (input-output linearization (IOL) or state-space linearization (SSL)) and the major focus of the paper.

Biological Reactors

Feedback linearizing controllers have been developed for several types of biological reactor models. The most commonly used model describes the growth of a single cell population from a single, rate-limiting substrate in a continuous stirred tank reactor [4]:

$$\begin{aligned}
\dot{X} &= \mu(X,S)X - DX \\
\dot{S} &= -\sigma(X,S)X - D(S_f - S)
\end{aligned} \tag{4.141}$$

Typically, the specific growth rate μ is modeled by a Monod relation,

$$\mu(X,S) = \mu(S) = \frac{\mu_m}{K_m + S} \tag{4.142}$$

and a constant yield expression is used to describe the specific substrate consumption rate σ:

Table 4.1: Applications to Chemical Reactors

References	Approach	Focus
[96]	IOL	basic controller design
[1]	IOL	global feedback stabilization
[19, 40]	IOL	disturbance decoupling
[42]	IOL	input-output decoupling
[68]	IOL	general nonlinear processes
[74, 103]	IOL	time delay compensation
[8, 67, 92]	IOL	constrained processes
[3, 12, 15, 101]	IOL	robust controller design
[151, 154]	IOL	discrete-time models
[23, 51, 72, 136]	IOL	internal model control
[50, 143]	IOL	approximate feedback linearization
[48, 97, 169]	IOL	non-minimum phase compensation
[13, 21]	IOL	adaptive feedback linearization
[108, 125]	IOL	differential/algebraic models
[41, 98, 109, 170]	IOL	output feedback control
[11, 77, 90]	SSL	basic controller design
[7]	SSL	global feedback stabilization
[30, 32, 33]	SSL	feedforward/feedback control
[80]	SSL	time delay compensation
[49, 91]	SSL	robust controller design
[50]	SSL	approximate feedback linearization
[117]	SSL	output feedback control

$$\sigma(X,S) = \frac{1}{Y_{X/S}}\mu(S) \tag{4.143}$$

Bioreactor models of this form can exhibit significant static and dynamic nonlinear behavior [70]. The dilution rate D or feed substrate concentration S_f is usually employed as the manipulated input, while the cell concentration X or substrate concentration S is often chosen as the controlled output. A summary of applications of feedback linearization to bioreactor models is presented in Table 4.2.

Other Processes

Several other types of processes can benefit from feedback linearizing control as a result of their strongly nonlinear behavior. Important examples of such processes include polymerization reactors, high purity distillation columns, and weakly buffered pH neutralization systems. A summary of applications of feedback linearization to models of these processes is shown in Table 4.3.

Experimental Studies

As compared to simulation studies, relatively few experimental applications of feedback linearizing control have been presented. However, the number of experimental studies has increased significantly over the past

Table 4.2: Applications to Biological Reactors

References	Approach	Focus
[20, 38, 71]	IOL	basic controller design
[140]	IOL	input-output decoupling
[70]	IOL	internal model control
[71, 116]	IOL	approximate feedback linearization
[45]	IOL	non-minimum phase compensation
[46]	IOL	adaptive feedback linearization
[52, 71, 79]	SSL	basic controller design
[78]	SSL	multiple-input processes

Table 4.3: Applications to Other Processes

References	Process	Approach
[34, 36, 61, 126, 144]	distillation column	IOL
[6]	distillation column	SSL
[62, 112]	evaporator	IOL
[147]	heat exchanger	IOL
[5]	heat exchanger	SSL
[69, 75, 107, 137, 166, 168]	pH neutralization	IOL
[2, 43, 94, 102, 104, 124, 165]	polymerization reactor	IOL
[9, 10]	polymerization reactor	SSL
[114, 135]	storage tank	IOL
[141]	supercritical extractor	IOL

several years. As shown in Table 4.4, input-output linearization techniques have been applied to a variety of processes including biological reactors, distillation columns, pH neutralization systems, and polymerization reactors. By contrast, we have found no experimental applications of the state-space linearization approach.

4.6 Case Studies

4.6.1 Continuous Stirred Tank Reactors in Series

Process Model

The process consists of two constant volume reactors in which an irreversible, exothermic reaction A → B occurs. The effluent stream from the first reactor serves as the feed stream for the second reactor. The reactors are cooled by a single coolant stream flowing cocurrently with the reaction stream. Neglecting coolant dynamics, the process model consists of four nonlinear ordinary differential equations [68]:

$$\begin{aligned}
\dot{C}_{A1} &= \frac{q}{V_1}(C_{Af} - C_{A1}) - k_0 C_{A1} \exp\left(-\frac{E}{RT_1}\right) \\
\dot{T}_1 &= \frac{q}{V_1}(T_f - T_1) + \frac{(-\Delta H) k_0 C_{A1}}{\rho C_p} \exp\left(-\frac{E}{RT_1}\right)
\end{aligned}$$

Table 4.4: Experimental Studies

Reference	Process	Focus
[139]	bioreactor	input-output decoupling
[142, 146]	bioreactor	adaptive input-output linearization
[18, 39]	distillation column	input-output linearization
[115]	distillation column	disturbance decoupling
[47]	distillation column	input-output decoupling
[167, 171]	pH neutralization	input-output linearization
[58, 73]	pH neutralization	adaptive input-output linearization
[152]	polymerization reactor	input-output linearization
[153]	polymerization reactor	input-output decoupling

Table 4.5: Nominal Parameters for the CSTR Model

Variable	Value	Variable	Value
q	100 L/min	k_0	7.2×10^{10} min^{-1}
C_{Af}	1 mol/L	$\frac{E}{R}$	1×10^4 K
T_f	350 K	$-\Delta H$	4.78×10^4 J/mol
T_{cf}	350 K	ρ, ρ_c	1000 g/L
V_1, V_2	100 L	C_p, C_{pc}	0.239 J/g·K
hA_1, hA_2	1.67×10^5 J/min·K		

$$
\begin{aligned}
& +\frac{\rho_c C_{pc}}{\rho C_p V_1} q_c \left[1 - \exp\left(-\frac{UA_1}{q_c \rho_c C_{pc}}\right)\right](T_{cf} - T_1) \qquad (4.144) \\
\dot{C}_{A2} &= \frac{q}{V_2}(C_{A1} - C_{A2}) - k_0 C_{A2} \exp\left(-\frac{E}{RT_2}\right) \\
\dot{T}_2 &= \frac{q}{V_2}(T_1 - T_2) + \frac{(-\Delta H) k_0 C_{A2}}{\rho C_p} \exp\left(-\frac{E}{RT_2}\right) + \frac{\rho_c C_{pc}}{\rho C_p V_2} q_c \\
& \times \left[1 - \exp\left(-\frac{UA_2}{q_c \rho_c C_{pc}}\right)\right]\left[T_1 - T_2 + \exp\left(-\frac{UA_1}{q_c \rho_c C_{pc}}\right)(T_{cf} - T_1)\right]
\end{aligned}
$$

We have used standard notation [162] where the subscripts 1, 2, c, and f denote the first reactor, second reactor, coolant stream, and feed stream, respectively. Nominal parameters for the model are shown in Table 4.5. The objective is to control the effluent composition from the second tank (C_{A2}) by manipulating the coolant flow rate (q_c). Note that the model is not control affine when $u = q_c$. If the state variables and controlled output are defined as,

$$
x^T = \begin{bmatrix} C_{A1} & T_1 & C_{A2} & T_2 \end{bmatrix}, \qquad y = C_{A2} \qquad (4.145)
$$

the model has the form of the general nonlinear system (4.69). As a result, the nonlinear controller design is based on the input-output linearization techniques discussed in Section 5.4.1.

Figure 4.1: Open-loop response for ± 10% change in q_c [68].

Controller Design

For the purpose of comparison, three controllers are designed: a conventional proportional-integral (PI) controller; a static input-output linearizing controller; and a dynamic input-output linearizing controller. The PI controller is tuned for setpoint changes, yielding k_c = 350 L^2/mol·min and τ_I = 0.25 min. The nonlinear controllers are designed by assuming that all four state variables are available for feedback. The static nonlinear controller design is based on the original nonlinear model (4.144), which has relative degree $r = 2$. The control moves of the static controller are generated by solving a nonlinear algebraic equation of the form (4.76) at each sampling instant. The dynamic nonlinear controller is designed using an extended model of the form (4.78), which has relative degree $r = 3$ in this case. For both nonlinear controllers, the transformed input v is chosen to yield a nominal closed-loop transfer function of the form (4.23) where ϵ = 0.25 min. The expressions for the two nonlinear controllers are rather complicated, and therefore they are not presented here.

Simulation Results

The open-loop composition responses in Figure 5.1 demonstrate that the model exhibits highly nonlinear behavior. The disturbance rejection performance of the three controllers for ± 5% unmeasured disturbances in the feed composition (C_{Af}) is shown in Figure 5.2. The nonlinear controllers cannot completely decouple the disturbance from the output because C_{Af} has relative degree $\rho = 1$. The static nonlinear controller provides excellent performance and is clearly superior to the other controllers for both disturbances. Note that the PI controller provides more effective rejection of the + 5% disturbance than does the dynamic nonlinear controller.

The controllers are compared for unmeasured feed temperature (T_f) disturbances of ± 25 K in Figure 5.3. Because T_f has relative degree $\rho = 3$, the static nonlinear controller provides perfect disturbance decoupling and therefore it outperforms the other controllers. The dynamic nonlinear controller provides

Figure 4.2: Closed-loop response for ± 5% C_{Af} disturbance [68].

Figure 4.3: Closed-loop response for $\pm$ 25% q disturbance [68].

Table 4.6: Nominal Operating Conditions for the Fermentor Model

Variable	Value	Variable	Value
$Y_{X/S}$	0.4 g/g	α	2.2 g/g
β	0.2 h^{-1}	μ_m	0.48 h^{-1}
P_m	50 g/L	K_m	1.2 g/L
K_i	22 g/L	S_f	20 g/L
D	0.202 h^{-1}	X	6.0 g/L
S	5.0 g/L	P	19.14 g/L

vastly superior performance as compared to the PI controller for the – 25 K disturbance. The responses of the two controllers are comparable for the + 25 K disturbance. These results indicate that a disturbance can be rejected most effectively if the difference between the relative degrees of the manipulated input and the disturbance is small. This difference is always smaller for the static controller than for the dynamic controller. Consequently, we propose that the manipulated input and controlled output of an input-output linearizing controller should be chosen such that the relative degree r is *minimized*. This proposition provides another argument against the state-space linearization approach, which results in a *maximal* relative degree.

4.6.2 Continuous Fermentor

Process Model

The process consists of a constant volume reactor in which a single, rate limiting substrate promotes biomass growth and product formation. By assuming constant yields, a process model with three nonlinear ordinary differential equations can be obtained [4],

$$\begin{aligned}
\dot{X} &= -DX + \mu(S,P)X \\
\dot{S} &= D(S_f - S) - \frac{1}{Y_{X/S}}\mu(S,P)X \\
\dot{P} &= -DP + [\alpha\mu(S,P) + \beta]\,X
\end{aligned} \tag{4.146}$$

where: X, S, and P are the biomass, substrate, and product concentrations, respectively; D is the dilution rate; S_f is the feed substrate concentration; and $Y_{X/S}$, α, and β are yield parameters. The specific growth rate μ is modeled as,

$$\mu(S,P) = \frac{\mu_m\left(1 - \frac{P}{P_m}\right)S}{K_m + S + \frac{S^2}{K_i}} \tag{4.147}$$

where: μ_m is the maximum specific growth rate; and P_m, K_m, and K_i are constant parameters. Nominal operating conditions are shown in Table 4.6.

The control objective in many continuous fermentations is to maximize the steady-state biomass production. This can be a difficult task since parameters such as the maximum specific growth rate μ_m and cell-mass yield $Y_{X/S}$ may exhibit significant time-varying behavior. It can be shown [70] that near optimal steady-state performance can be achieved by manipulating the dilution rate D and regulating the biomass concentration X at a constant value. A nonlinear model of the form (4.1) can be obtained by choosing:

$$u = D, \quad x^T = \begin{bmatrix} X & S & P \end{bmatrix}, \quad y = X \tag{4.148}$$

Controller Design

Three controllers are designed: a PI controller; an input-output linearizing controller based on full state feedback; and a nonlinear IMC controller that requires only a biomass concentration measurement. The PI controller parameters are determined initially by applying IMC tuning rules [129] to a first-order linear model obtained from the open-loop responses in Figure 5.4. The IMC closed-loop time constant is chosen as one-third the open-loop time constant. The controller parameters are fine-tuned for setpoint responses, yielding $k_c = 0.07$ L/g·h and $\tau_I = 4.5$ h.

The nonlinear controllers are designed by noting that the relative degree $r = 1$ since $L_g h(x) = -x_1$. The input output linearizing controller is synthesized as described in Section 5.2.2:

$$\begin{aligned} u &= \frac{v - \mu(x_2, x_3)x_1}{-x_1} \\ v &= \frac{2}{\epsilon}\left[y_{sp} - x_1\right] + \frac{1}{\epsilon^2}\int_0^t \left[y_{sp} - x_1\right] d\tau \end{aligned} \tag{4.149}$$

Using the design procedure in Section 5.5.1, the following nonlinear IMC controller is obtained,

$$\begin{aligned} u &= \frac{\dot{v} - \mu(\tilde{x}_2, \tilde{x}_3)\tilde{x}_1}{-\tilde{x}_1} \\ \dot{v} &= -\frac{1}{\epsilon}\tilde{x}_1 + \frac{1}{\epsilon}(y_{sp} - x_1 + \tilde{x}_1) \end{aligned} \tag{4.150}$$

where the tilde represents a variable obtained from the process model. Both nonlinear controllers are tuned with $\epsilon = 1$ h, which is approximately one-third the open-loop time constant for the – 10% dilution rate change in Figure 5.4.

Simulation Results

The open-loop biomass concentration response shown in Figure 5.4 demonstrate that the fermentor exhibits significant static and dynamic nonlinear behavior. In Figure 5.5, the three controllers are compared for an unmeasured step disturbance of – 12.5% in the maximum specific growth rate (μ_m). The input-output linearizing cannot provide complete decoupling because the disturbance relative degree $\rho = 1$ and μ_m is unmeasured. However, the linearizing controller yields superior regulatory performance as compared to the PI and nonlinear IMC controllers. This result is expected since the linearizing controller has access to the entire state vector. On the other hand, the nonlinear IMC controller yields vastly improved disturbance rejection as compared to the PI controller despite the fact that both controllers only use a measurement of the biomass concentration.

The controllers are compared for a – 20% step disturbance in the cell-mass yield ($Y_{X/S}$) in Figure 5.6. The input-output linearizing controller yields perfect disturbance decoupling because it has access to the entire state vector and $\rho = 2$. By contrast, the nonlinear IMC controller uses state estimates from the

Figure 4.4: Open-loop response for ± 10% change in D [70].

Figure 4.5: Closed-loop response for – 12.5% disturbance in μ_m [70].

model and therefore it cannot provide exacte decoupling. However, the IMC controller yields superior performance as compared to the PI controller. The estimated state variables used by the IMC controller also are shown in Figure 5.6. Despite the significant errors caused by the unmeasured disturbance, the IMC controller provides satisfactory control.

Figure 5.7 shows the responses of the controllers responses for the same $Y_{X/S}$ disturbance when there is a stuctural error in the specific growth rate. The growth rate used for nonlinear controller design is a simple Monod expression:

$$\mu = \frac{\mu_m}{K_m + S} \tag{4.151}$$

The PI controller response is identical to that shown in Figure 5.6 since the modeling error does not affect the process. By contrast, the nonlinear controller responses are changed significantly. The response of the nonlinear IMC controller is actually improved, while the input-output linearizing controller is no longer able to provide complete decoupling. In fact, the IMC controller outperforms the linearizing controller in this case. Additional simulation results [70] show that the nonlinear IMC controller is superior to the PI controller and compares favorably to the input-output linearizing controller based on full-state feedback.

4.6.3 pH Neutralization System

Process Model

The process consists of an acid (HNO_3) stream, buffer ($NaHCO_3$) stream, and base ($NaOH$) stream that are mixed in a stirred tank. The chemical equilibria is modeled by introducing two reaction invariants [58] for each inlet stream,

$$\begin{aligned} W_{ai} &= [H^+]_i - [OH^-]_i - [HCO_3^-]_i - 2[CO_3^=]_i \\ W_{bi} &= [H_2CO_3]_i + [HCO_3^-]_i + [CO_3^=]_i \end{aligned} \tag{4.152}$$

where $i = 1$ for the acid stream, $i = 2$ for the buffer stream, and $i = 3$ for the base stream. By combining mass balances on each of the ionic species in the system, the following differential equations for the effluent reaction invariants can be derived [60]:

$$\begin{aligned} \dot{W}_{a4} &= \frac{1}{Ah}(W_{a1} - W_{a4})q_1 + \frac{1}{Ah}(W_{a2} - W_{a4})q_2 + \frac{1}{Ah}(W_{a3} - W_{a4})q_3 \\ \dot{W}_{b4} &= \frac{1}{Ah}(W_{b1} - W_{b4})q_1 + \frac{1}{Ah}(W_{b2} - W_{b4})q_2 + \frac{1}{Ah}(W_{b3} - W_{b4})q_3 \end{aligned} \tag{4.153}$$

where: q_1, q_2, and q_3 are the volumetric flow rates of the acid, buffer, and base streams, respectively; A is the cross-sectional area of the mixing tank; and h is the liquid level.

The effluent pH is determined from W_{a4} and W_{b4} using the following relation,

$$W_{a4} + 10^{pH-14} - 10^{-pH} + W_{b4}\frac{1 + 2 \times 10^{pH-pK_2}}{1 + 10^{pK_1-pH} + 10^{pH-pH_2}} = 0 \tag{4.154}$$

where pK_1 and pK_2 are the base-10 logarithms of the equilibrium constants associated with H_2CO_3 and HCO_3^- disassociation, respectively. Because the pH probe is located downstream from the mixing tank,

Figure 4.6: Closed-loop response for ± 20% $Y_{X/S}$ disturbance [70].

Figure 4.7: Closed-loop response with modeling error for ± 20% $Y_{X/S}$ disturbance [70].

Table 4.7: Nominal Operating Conditions for the pH System

Variable	Value	Variable	Value
W_{a1}	0.003 M	W_{b1}	0 M
W_{a2}	- 0.03 M	W_{b2}	0.03 M
W_{a3}	- 3.05 × 10^{-3} M	W_{b3}	5 × 10^{-5} M
K_{a1}	4.47 × 10^{-7}	K_{a2}	5.62 × 10^{-11}
A	207 cm^2	n	0.607
z	11.5 cm	q_1	16.6 ml/s
q_2	0.55 ml/s	q_3	15.6 ml/s
W_{a4}	- 4.32 × 10^{-4} M	W_{b4}	5.28 × 10^{-4} M
h	14.0 cm	pH	7.0

there is an unmodeled time delay of approximately 10 s associated with the pH measurement. The liquid level is modeled as,

$$\dot{h} = \frac{1}{A}\left[q_1 + q_2 + q_3 - C_v(h+z)^n\right] \tag{4.155}$$

where C_v is the valve coefficient, n is the valve exponent, and z is the vertical distance between the bottom of the mixing tank and the outlet of the effluent stream. On-line measurements of the liquid level and effluent pH are available, while the reaction invariants must be estimated. Nominal operating conditions are shown in Table 4.7.

The objective is to control the pH despite unmeasured acid and buffer flow rate disturbances by manipulating the base flow rate. It is important to have an accurate estimate of the buffer flow rate because it determines the buffering capacity of the systems. A nonlinear state-space model is obtained by defining:

$$x^T = \begin{bmatrix} W_{a4} & W_{b4} & h \end{bmatrix}, \qquad u = q_3, \qquad y = pH, \qquad d = q_2 \tag{4.156}$$

The resulting model has the form,

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + p(x)d \\ c(x,y) &= 0 \end{aligned} \tag{4.157}$$

where the definitions of the functions $f(x)$, $g(x)$, $p(x)$, and $c(x,y)$ follow directly from the process model. Note that the output equation is an *implicit* function of the output (y); that is, a closed form representation $y = h(x)$ cannot be determined. As a result, a modified technique is proposed for the design of the input-output linearizing controller.

Input-Output Linearizing Controller Design

The input-output linearizing controller is designed by taking the time derivative of the output equation and rearranging,

$$\dot{y} = -c_y^{-1}(x,y)c_x(y)[f(x) + g(x)u + p(x)d] \tag{4.158}$$

where:

$$\begin{aligned} c_x^T(y) &= \begin{bmatrix} 1 & \frac{1+2\times10^{y-pK_2}}{1+10^{pK_1-y}+10^{y-pK_2}} & 0 \end{bmatrix} \\ c_y(x,y) &= \ln 10\left[10^{y-14} + 10^{-y}\right] + \\ & \quad \ln 10\left[x_2 \frac{10^{pK_1-y} + 10^{y-pK_2} + 4\left(10^{pK_1-y}\right)\left(10^{y-pK_2}\right)}{(1+10^{pK_1-y}+10^{y-pK_2})^2}\right] \end{aligned} \tag{4.159}$$

Because the term $c_y^{-1}(x,y)c_x(y)g(x)$ that multiplies u is non-zero for all operating points of interest, the model is said to have relative degree $r = 1$. The input-output linearizing control law is:

$$u = \frac{v + c_y^{-1}(x,y)c_x(y)[f(x) + p(x)d]}{-c_y^{-1}(x,y)c_x(y)g(x)} \tag{4.160}$$

The transformed input v is chosen as,

$$v = 2\epsilon^{-1}[y_{sp} - y] + \epsilon^{-2}\int_0^t [y_{sp} - y]d\tau \tag{4.161}$$

where ϵ is the controller tuning parameter. In the absence of plant/model mismatch, the control law yields the closed-loop transfer function (4.23) with $r = 1$. The controller is tuned with $\epsilon = 1$ min, which is approximately one-half the time constant for the open-loop responses in Figure 5.8. To facilitate experimental implementation, the control law is discretized with $\Delta t = 15s$ [73].

Table 4.8: Standard Inlet Flow Changes

Time (min)	q_1 (ml/s)	q_2 (ml/s)	q_3 (ml/s)
0	16.6	0.55	15.6
30	15.1	1.2	13.6
60	18.1	2.0	17.6
90	16.6	1.0	15.6
120	–	0	–
150	–	0.55	–

State and Parameter Estimation

The input-output linearizing controller requires full state feedback. In practice, the reaction invariants cannot be measured and therefore they must be estimated from available on-line measurements. A closed-loop nonlinear observer with exponentially stable error dynamics (see Chapter 7) does not exist since the Jacobian linearization of the model is unobservable at every equilibrium point [73]. As an alternative, an open-loop nonlinear observer that does not require observability is employed. Because the differential equations for the reaction invariants are decoupled, the invariants can be estimated sequentially [73]. Invariant W_{b4} is estimated in an open-loop fashion. An estimate of W_{a4} is generated from the estimated W_{b4} and the measured pH using the output equation.

The proposed controller does not explicitly account for buffering changes. As shown below, unstable behavior is obtained if the buffer flow rate drops significantly below the nominal value. Under these conditions, the process gain increases dramatically and some type of on-line adaptation is necessary to achieve satisfactory performance. A straightforward approach is to treat the buffer flow rate as an unknown parameter that is estimated with a recursive least-squares algorithm. The parameter estimator receives measurements of the pH and level, as well as reaction invariant estimates from the open-loop observer. A detailed derivation of the estimator is presented elsewhere [73].

Experimental Results

Open-loop pH responses for the sequence of base and buffer flow rate changes in Table 4.8 are shown in Figure 5.8. Note that the actual run time is plotted and therefore the plots do not begin with zero time. The response in Figure 5.8(a) shows that the system exhibits significant static nonlinearities with respect to manipulated input changes. The pH response for buffer flow rate changes is shown in Figure 5.8(b). Significant static nonlinear behavior is observed, especially when $q_2 \to 0$ ml/s. The pH response for base flow rate changes is similar to that obtained from the process model, while significant deviations are observed between the simulated and experimental responses for buffer flow rate changes [65].

As discussed in [73], non-adaptive and adaptive versions of the input-output linearizing controller easily outperform a PI controller. However, Figure 5.9 shows that the non-adaptive controller exhibits unstable behavior for low buffering conditions. The sequence of buffer flow rate disturbances is the same as that in Table 4.8 except that the flow rate is only reduced to 0.2 ml/s at $t = 120$ min. The non-adaptive controller produces a highly oscillatory response as a result of the $q_2 = 0.2$ ml/s disturbance at $t = 215$ min. This unstable behavior is attributable to poor estimates of the reaction invariants at low buffering conditions.

The performance of the adaptive nonlinear controller for the same sequence of buffer flow rate disturbances is shown in Figure 5.10. The controller is able to provide excellent control over a wide range of buffering conditions. Unlike the non-adaptive controller (Figure 5.9), the adaptive controller is able to maintain the system at the setpoint even when $q_2 \rightarrow 0.2$ ml/s. The estimated buffer flow rate produced by the adaptive controller also is shown in Figure 5.10. At steady-state conditions, the estimation error is less than 15% of the actual value. Improved knowledge of the buffering capacity results in more accurate estimates of the reaction invariants, which ultimately leads to superior closed-loop performance as compared to the non-adaptive case.

The performance of the adaptive nonlinear controller for the sequence of acid flow rate disturbances in Table 4.8 is shown in Figure 5.11. Acid flow rate disturbances represent a robustness test since the parameter estimator is designed only to account for buffering changes. Note that the estimated buffer flow rate used by the controller is set equal to the nominal value (0.55 ml/s) if the value produced by the estimator is sufficiently negative. Despite using poor estimates of the buffer flow rate and the reaction invariants, the adaptive controller provides superior pH responses as compared to the PI and non-adaptive nonlinear controllers (not shown).

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