

MONOCHROMATIC BOXES IN COLORED GRIDS*

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Abstract. A d -dimensional *grid* is a set of the form $L = [a_1] \times \cdots \times [a_d]$, where $[t] = \{1, \dots, t\}$ and a_j is a positive integer for $j \in [d]$. A d -dimensional *box* is a set of the form $\{x_1, y_1\} \times \cdots \times \{x_d, y_d\}$ for some integers x_j, y_j with $j \in [d]$, where $x_j \neq y_j$ for each j . We give conditions on the set of d -tuples (a_1, \dots, a_d) so that, for every coloring $f: L \rightarrow [c]$, the grid $L = [a_1] \times \cdots \times [a_d]$ contains a box on which f is constant. In particular, we analyze the set of grids that are minimal with respect to this property. We show that, for $d \geq 3$, this set has size $O(c^{(17 \cdot 3^{d-3} - 1)/2})$, and all its elements have volume $O(c^{(3^d - 1)/2})$ as $c \rightarrow \infty$.

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1. Introduction. Let $[t]$ denote the set $\{1, \dots, t\}$ for any positive integer t . A d -dimensional *grid* is a set $L = [a_1] \times \cdots \times [a_d]$. For ease of notation, we write $[a_1, \dots, a_d]$ for $[a_1] \times \cdots \times [a_d]$. The “volume” of L is $\prod_{i=1}^d a_i$. A d -dimensional *box* is a set of 2^d points of the form

$$\{x_1, y_1\} \times \cdots \times \{x_d, y_d\}$$

with x_j and y_j integers so that $x_j \neq y_j$ for each $j \in [d]$. A grid L is (c, t) -guaranteed if, for all colorings $f: L \rightarrow [c]$, there are at least t distinct monochromatic boxes in L , i.e., boxes $B_j \subseteq L$, $j \in [t]$, so that $|f(B_j)| = 1$. We say that a grid is c -uncolorable to mean that it is $(c, 1)$ -guaranteed. If L is not c -uncolorable, we say it is c -colorable. If, for all $i \in [d]$, $b_i \geq a_i$, then $[b_1, \dots, b_d]$ is c -uncolorable if $[a_1, \dots, a_d]$ is. Hence, a full description of the set of c -uncolorable grids is given by its minimal elements with respect to this partial order. Call the set of minimal c -uncolorable grids $\mathcal{O}(c, d)$ the *obstruction set* for c colors in dimension d . It is well known from poset theory that this set is always finite. We focus our attention on *monotone* grids, i.e., those grids for which $a_1 \leq \cdots \leq a_d$, since being c -uncolorable (or (c, t) -guaranteed) is invariant under permutations of the a_j .

Our main contribution is a rough description of the obstruction set for c colors in d dimensions. In the language of hypergraphs, we provide bounds on the multiset of partite sizes of a complete multipartite hypergraph G , every c -coloring of which admits a monochromatic hypergraph H , for the smallest nontrivial choice of H : $K_d(2)$. A specialization gives bounds on the multipartite Ramsey number of $H = K_d(2)$, defined to be the smallest N so that every c -coloring of the complete balanced d -partite d -uniform hypergraph with partite sets of size N contains a monochromatic H . The $d = 2$ (un-

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balanced) case is considered in greater detail in [7], and closely related questions are treated in [4] and [9] (albeit in the language of graph theory).

The subject of unavoidable configurations in grids has played a prominent role in combinatorics, not least because it offers many natural generalizations of the celebrated van der Waerden's and Szemerédi's theorems on arithmetic progressions. Often the configuration of concern is a grid which is further required to have the same "spacing" in each direction; see, for example, [11], where the author shows that every coloring of \mathbb{N}^2 contains a monochromatic *square*. Indeed, the famous Gallai–Witt theorem (q.v. [8]) states that every configuration is unavoidable, although no known proofs offer reasonable bounds.

THEOREM 1 (Gallai–Witt theorem). *Let m, n, k be positive integers. If the points of \mathbb{Z}^n are colored with k colors, and A is any m -element subset of \mathbb{Z}^n , then there is a monochromatic subset of \mathbb{Z}^n that is homothetic to (i.e., a dilation and shift of) A .*

The Gallai–Witt theorem ensures, in particular, that the quantity $N(c, d)$, defined to be the least N so that $[N]^d$ is c -uncolorable, is finite. An analysis of $N(c, d)$ (and some related quantities) appears in the manuscript by Agnarsson, Doerr, and Schoen [1]. Bounds for some configurations in two dimensions are found in [3].

Another reason for interest in unavoidable configurations in grids is the connection with Ramsey theory. Let G be the complete d -partite d -uniform hypergraph on partite sets $[a_1], \dots, [a_d]$, and let $K_d(2)$ denote such a complete d -partite hypergraph with exactly two vertices in each partite set. Any point in the grid $L = [a_1, \dots, a_d]$ corresponds to a hyperedge in G ; a box in L corresponds to a $K_d(2)$ subhypergraph of G . To say that L is c -uncolorable is equivalent to the statement that every c -coloring of G contains a monochromatic $K_d(2)$. In [5] and [6], the authors prove bounds on multipartite Ramsey numbers of various (2-uniform) graphs. Little is known about multipartite Ramsey numbers of hypergraphs, although the bipartite Ramsey numbers of graphs are well studied. (One notable appearance of (classical) Ramsey numbers for unbalanced multipartite hypergraphs is [10].)

In the next section, we show that any grid of sufficiently small volume (approximately c^{2^d-1}) is c -colorable. The following section shows that the analysis is tight: there are grids of this volume which are c -uncolorable. Not all grids of sufficiently large volume are c -guaranteed, although section 4 demonstrates that any grid, all of whose lower-dimensional subgrids are sufficiently voluminous, is indeed c -guaranteed. Section 5 gives a tight upper bound on the volume of minimally c -uncolorable grids, i.e., elements of the obstruction set. Section 6 then addresses the question of how many obstructions there are. Finally, as mentioned above, section 7 considers the case of $c = 2$ and $d = 3$, where some computational questions arise. This complements work of the second two authors [7] for $d = 2$ and $2 \leq c \leq 4$.

Throughout the present manuscript, we write $f = O(g)$ as $t \rightarrow \infty$ ($f = \Omega(g)$ as $t \rightarrow \infty$), for nonnegative functions $f, g: \mathbb{R} \rightarrow \mathbb{R}$ and some parameter t , to mean that there exist nonnegative constants C and N (possibly depending on d) such that $f(x) \leq Cg(x)$ (respectively, $f(x) \geq Cg(x)$) whenever $t > N$. The expression $f = \Theta(g)$ as $t \rightarrow \infty$ means that $f = O(g)$ as $t \rightarrow \infty$ and $g = \Omega(f)$ as $t \rightarrow \infty$, and the expression $f = o(g)$ as $t \rightarrow \infty$ means that, for all $\epsilon > 0$, there exists an N_ϵ (possibly depending on d) so that $f(x) \leq \epsilon g(x)$ whenever $t > N_\epsilon$.

2. All small grids are c -colorable. Define $V(c, d)$ to be the largest integer V so that every d -dimensional grid L with volume at most V is c -colorable. Below we show that $V(c, d)$ is $\Theta(c^{2^d-1})$ as $c \rightarrow \infty$. Lower and upper bounds were proved for the equi-

lateral case by Agnarsson, Doerr, and Schoen [1]. The proof of Theorem 2 is quite similar to that of Theorem 2.2 in [1], although we obtain a slightly stronger lower bound in the equilateral case. The upper bound is given by Corollary 8.

THEOREM 2. $V(c, d) > c^{2^d-1}/(e2^d)$, where $e = 2.718\dots$ is the base of the natural logarithm.

Proof. We apply the Lovász local lemma (see, e.g., [2]), which states the following. Suppose that A_1, \dots, A_t are events in some probability space, each of probability at most p . Let G be a “dependency” graph with vertex set $\{A_i\}_{i=1}^t$, i.e., a graph so that, whenever a set S of vertices induces no edges in G , then S is a mutually independent family of events. Then $\mathbf{P}(\bigwedge_{i=1}^t \bar{A}_i) > 0$ if $ep(\Delta + 1) \leq 1$, where $\Delta = \Delta(G)$ is the maximum degree of G .

Now suppose that $L = [a_1, \dots, a_d]$ is a grid of volume V and that we color the points of L uniformly at random from $[c]$. Enumerate all boxes in L as B_1, \dots, B_t . Define A_i to be the event that B_i is monochromatic in this random coloring. We may take G to have an edge between A_i and A_j whenever $B_i \cap B_j \neq \emptyset$ since any family of events A_i corresponding to mutually disjoint boxes is mutually independent. The degree of a vertex A_i is then the number of boxes B_j , $j \neq i$, which intersect B_i . Since we may specify the list of all such boxes by choosing one of the 2^d points of B_i and then choosing the d coordinates of its antipodal point, $\deg_G(A_i)$ is at most

$$2^d \prod_{i=1}^d (a_i - 1) - 1 < 2^d \prod_{i=1}^d a_i - 1 = 2^d V - 1.$$

(The outermost -1 here reflects the fact that B_i may be excluded among these choices.) The probability of each A_i is the same: $p = c^{-2^d+1}$. Therefore,

$$ep(\Delta + 1) < ec^{-2^d+1}2^d V$$

which is ≤ 1 whenever $V \leq c^{2^d-1}/(e2^d)$.

3. Some large grids are c -uncolorable. Recall the definition of (c, t) -guaranteed given in the first paragraph.

THEOREM 3. Fix c, d , define $L = [a_1, \dots, a_d]$, and let $M = \prod_i \binom{a_i}{2}$ denote the total number of boxes in L . Then L is $(c, M(1 + o(1))/c^{2^d-1})$ -guaranteed as $\min\{a_1, \dots, a_d\} \rightarrow \infty$.

Theorem 3 follows quickly from the next lemma, whose extra strength is needed later.

LEMMA 4. Suppose $c \geq 1$. For $d \geq 1$ and integers $a_1, \dots, a_d \geq 2$, let $M = \prod_{i=1}^d \binom{a_i}{2}$. Define Δ_j , $0 \leq j \leq d$, recursively as follows:

$$\Delta_0 = 1, \quad \Delta_j = \Delta_{j-1}^2 \left(1 - \frac{\frac{c^{2^j-1}}{\Delta_{j-1}} - 1}{a_j - 1}\right).$$

Then the grid $L = [a_1, \dots, a_d]$ is $(c, M\Delta_d/c^{2^d-1})$ -guaranteed provided $\Delta_1, \dots, \Delta_d > 0$.

Proof. We proceed inductively. Suppose $d = 1$, let $f: [a_1] \rightarrow [c]$ be a c -coloring, and define

$$\gamma_i = |f^{-1}(i)|$$

to be the number of points colored i , $1 \leq i \leq c$. Then the number N of monochromatic boxes in f is exactly

$$N = \sum_{i=1}^c \binom{\gamma_i}{2} = \frac{1}{2} \cdot \sum_{i=1}^c (\gamma_i^2 - \gamma_i) = \frac{1}{2} \cdot \left(\sum_{i=1}^c \gamma_i^2 - a_1 \right).$$

Applying the Cauchy–Schwarz inequality,

$$\begin{aligned} N &\geq \frac{\left(\sum_{i=1}^c \gamma_i \right)^2}{2c} - \frac{a_1}{2} = \frac{a_1^2}{2c} - \frac{a_1}{2} \\ &= \frac{a_1(a_1 - c)}{2c} = \frac{1}{c} \binom{a_1}{2} \frac{a_1 - c}{a_1 - 1} = \frac{1}{c} \binom{a_1}{2} \Delta_1. \end{aligned}$$

Now suppose the statement of Lemma 4 is true for dimensions less than $d + 1$, and consider a coloring $f: [a_1, \dots, a_{d+1}] \rightarrow [c]$. Consider the a_{d+1} colorings f_j of the d -dimensional grid $[a_1, \dots, a_d]$ induced by setting the last coordinate to j , i.e.,

$$f_j(x_1, \dots, x_d) = f(x_1, \dots, x_d, j).$$

Let $\gamma_i(B)$, for a box $B \subset [a_1, \dots, a_d]$ and $i \in [c]$, denote the number of j 's so that $f_{j|B} \equiv i$. Then the number N of monochromatic $(d + 1)$ -dimensional boxes in f is

$$\begin{aligned} N &= \sum_i \sum_B \binom{\gamma_i(B)}{2} \\ &= \frac{1}{2} \cdot \sum_i \sum_B (\gamma_i(B)^2 - \gamma_i(B)) \\ &\geq \frac{\left(\sum_B \sum_i \gamma_i(B) \right)^2}{2Mc} - \frac{1}{2} \cdot \sum_B \sum_i \gamma_i(B) \\ &= \frac{\left(\sum_B \sum_i \gamma_i(B) \right)^2 - Mc \sum_B \sum_i \gamma_i(B)}{2Mc}, \end{aligned}$$

where $M = \prod_{i=1}^d \binom{a_i}{2}$. Since, by the inductive hypothesis, f_j induces at least $M\Delta_d/c^{2^d-1}$ monochromatic boxes,

$$\sum_i \sum_B \gamma_i(B) \geq \frac{a_{d+1} M \Delta_d}{c^{2^d-1}}$$

so that

$$\begin{aligned}
N &\geq \frac{a_{d+1}^2 M^2 \Delta_d^2 / c^{2^{d+1}-2} - a_{d+1} M^2 \Delta_d c / c^{2^d-1}}{2Mc} \\
&= \frac{a_{d+1} M (a_{d+1} \Delta_d^2 - c^{2^d} \Delta_d)}{2c^{2^{d+1}-1}} \\
&= \frac{M}{c^{2^{d+1}-1}} \binom{a_{d+1}}{1} \frac{a_{d+1} \Delta_d^2 - c^{2^d} \Delta_d}{a_{d+1} - 1} \\
&= \frac{\prod_{i=1}^{d+1} \binom{a_i}{2}}{c^{2^{d+1}-1}} \cdot \Delta_d^2 \left(\frac{a_{d+1} - c^{2^d} / \Delta_d}{a_{d+1} - 1} \right) \\
&= \frac{\prod_{i=1}^{d+1} \binom{a_i}{2} \Delta_{d+1}}{c^{2^{d+1}-1}}.
\end{aligned}$$

Proof of Theorem 3. Fix $c, d \geq 1$. It is clear by induction on j that, for all $1 \leq j \leq d$, $\Delta_j = 1 + o(1)$ as $\min\{a_1, \dots, a_d\} \rightarrow \infty$, and so, in particular, $\Delta_j > 0$ if $\min\{a_1, \dots, a_d\}$ is large enough.

Note that, in the notation of Lemma 4, if $\Delta_1, \dots, \Delta_d > 0$, then $[a_1, \dots, a_d]$ is not c -colorable. Therefore we may conclude the following.

COROLLARY 5. *In the notation of Lemma 4, let Γ_j , $0 \leq j \leq d$, be given by the recurrence*

$$\begin{aligned}
\Gamma_0 &= 1, \\
\Gamma_j &= \Gamma_{j-1}^2 \left(1 - \frac{c^{2^{j-1}} / \Gamma_{j-1}}{a_j - 1} \right) = \Gamma_{j-1} \left(\Gamma_{j-1} - \frac{c^{2^{j-1}}}{a_j - 1} \right).
\end{aligned}$$

If $\Gamma_1, \dots, \Gamma_d > 0$, then $[a_1, \dots, a_d]$ is c -uncolorable.

Proof. Assume $\Gamma_1, \dots, \Gamma_d > 0$. A routine induction shows that $\Gamma_j \leq \Delta_j$ for $0 \leq j \leq d$.

LEMMA 6. *In the notation of Lemma 4, let ε_j be given by the recurrence*

$$\varepsilon_0 = 0, \quad \varepsilon_j = 2\varepsilon_{j-1} + \frac{c^{2^{j-1}}}{a_j - 1}.$$

If $\varepsilon_d < 1$, then $[a_1, \dots, a_d]$ is c -uncolorable.

Proof. It is clear that $0 = \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \dots < \varepsilon_d$, and so, by assumption, $\varepsilon_i < 1$ for all $i \in [d]$. An induction on i shows that $\Gamma_i \geq 1 - \varepsilon_i$ for $0 \leq i \leq d$: This is clearly true for $i = 0$. Suppose $i < d$ and $\Gamma_i \geq 1 - \varepsilon_i$. Then setting $\eta := c^{2^i} / (a_{i+1} - 1)$ and noting that $\Gamma_i \geq 0$, we have

$$(1) \quad \Gamma_{i+1} = \Gamma_i(\Gamma_i - \eta) \geq \Gamma_i(1 - \varepsilon_i - \eta).$$

The term in the parentheses is positive:

$$1 - \varepsilon_i - \eta \geq 1 - 2\varepsilon_i - \eta = 1 - \varepsilon_{i+1} > 0$$

by assumption. Thus, continuing (1) and using the inductive hypothesis again,

$$\Gamma_i(1 - \varepsilon_i - \eta) \geq (1 - \varepsilon_i)(1 - \varepsilon_i - \eta) \geq 1 - 2\varepsilon_i - \eta = 1 - \varepsilon_{i+1}.$$

Motivated by the preceding lemma, for every $c, d \geq 1$ and grid $L = [a_1, \dots, a_d]$ with $a_i \geq 2$ for all $i \in [d]$, we define

$$\varepsilon_c(L) := \sum_{i=1}^d 2^{d-i} \frac{c^{2^{i-1}}}{a_i - 1}.$$

LEMMA 7. *If $L = [a_1, \dots, a_d]$ is c -colorable, then $\varepsilon_c(L) \geq 1$.*

Proof. We let $\varepsilon_j := \varepsilon_c([a_1, \dots, a_j]) = \sum_{i=1}^j 2^{j-i} c^{2^{i-1}} / (a_i - 1)$ for all j with $0 \leq j \leq d$, and notice that the ε_j satisfy the recurrence in Lemma 6.

COROLLARY 8. *For any fixed $d \geq 1$ and $c \geq 2$, if n is least such that $[n]^d$ is c -uncolorable, then $n < (d+2)c^{2^{d-1}}$. Furthermore,*

$$2^{-d}e^{-1} < \frac{V(c, d)}{c^{2^d-1}} < (d+2)^d 2^{d(d-1)/2}.$$

Proof. If we take $a_j = (d+1)2^{d-j}c^{2^{j-1}} + 1$ for all $1 \leq j \leq d$, then $\varepsilon_c(L) = d/(d+1) < 1$. The second result now follows from the fact that

$$\begin{aligned} \prod_{j=1}^d a_j &< \prod_{j=1}^d (d+2)2^{d-j}c^{2^{j-1}} \\ &= (d+2)^d 2^{\sum_{j=1}^d (d-j)} c^{\sum_{j=1}^d 2^{j-1}} \\ &= (d+2)^d 2^{\sum_{j=1}^{d-1} j} c^{\sum_{j=0}^{d-1} 2^j} \\ &= (d+2)^d 2^{d(d-1)/2} c^{2^d-1}. \end{aligned}$$

The first result follows by taking $n := a_d$.

4. Hereditarily large grids are c -uncolorable. It is possible for grids of arbitrarily large volume to be c -colorable. Indeed, one needs to have only one of the dimensions be at most c and then color the grid with this coordinate. However, if we require that each lower dimensional subgrid be sufficiently voluminous, then the whole grid is c -uncolorable. This statement is made precise by the following theorem.

THEOREM 9. *Fix $d > 0$, and define $C_j = (d2^d)^{\frac{3}{2}(3^{j-1}-1)}$ for $j \geq 1$. For all integers $c \geq 1$ and $1 \leq a_1 \leq a_2 \leq \dots \leq a_d$, if $\prod_{i=1}^j a_i > C_j c^{(3^{j-1}-1)/2}$ for all $j \in [d]$, then $[a_1, \dots, a_d]$ is c -uncolorable.*

We require a lemma and a bit of notation. If $L = [a_1, \dots, a_d]$ and $1 \leq j < d$, let L_j denote $[a_1, \dots, a_j]$, and let \bar{L}_j denote $[a_{j+1}, \dots, a_d]$. Note that if L is c -uncolorable, then L_j is as well. Indeed, if $f: L_j \rightarrow [c]$ is a c -coloring of L_j , then the function $g: L \rightarrow [c]$ defined by $g(x_1, \dots, x_d) = f(x_1, \dots, x_j)$ is a c -coloring of L . We will also make repeated use of the following easily verified fact: For every integer $j \geq 0$, $j \cdot 2^{j-1} \leq (3^j - 1)/2$ and $j \cdot 2^j + 1 \leq 3^j$.

LEMMA 10. *Let $c \geq 1$, let $L = [a_1, \dots, a_d]$ be a grid, and let $j \in [d-1]$. Define*

$$c' := c \cdot \prod_{i=1}^j \binom{a_i}{2} \leq 2^{-j} \cdot c \cdot \prod_{i=1}^j a_i^2.$$

If L_j is c -uncolorable and \bar{L}_j is c' -uncolorable, then L is c -uncolorable.

Proof. Assume that L_j is c -uncolorable and that \bar{L}_j is c' -uncolorable. Suppose that $f: L \rightarrow [c]$ is a c -coloring. Consider the coloring $g: \bar{L}_j \rightarrow [c']$ that assigns the pair (B, s)

to the point \mathbf{v} , B being an arbitrary choice of j -dimensional box which is colored monochromatically by $f_j: L_j \rightarrow [c]$, where $f_j(x_1, \dots, x_j) = f(x_1, \dots, x_j, \mathbf{v})$, and s being its color. (Note that L_j is c -uncolorable, so such a B always exists.) Then g is a c' -coloring because there are exactly c' many different (B, s) . Since \bar{L}_j is c' -uncolorable, g colors some $(d-j)$ -dimensional box B_1 monochromatically with color (B_2, s) . But then $B_2 \times B_1$ is a d -dimensional box monocolored by f with color s .

Proof of Theorem 9. The statement is clearly true when $d = 1$ since $C_1 = 1$. Suppose that $d > 1$ and that the statement is true for all $d' < d$. Let $L = [a_1, \dots, a_d]$ be a monotone grid satisfying the hypothesis of the theorem.

Case 1. $\varepsilon_c(L) < 1$. The result follows immediately from Lemma 7.

Case 2. $\varepsilon_c(L) \geq 1$. Then there is some $j \in [d]$ such that $2^{d-j}c^{2^{j-1}}/(a_j - 1) \geq 1/d$, i.e.,

$$a_j \leq d2^{d-j}c^{2^{j-1}} + 1 < d2^{d-j+1}c^{2^{j-1}}.$$

Since $t2^t \leq 3^t - 1$ for all integers $t \geq 1$,

$$\prod_{i=1}^j a_i \leq \prod_{i=1}^j a_j < d^j 2^{j(d-j+1)} c^{j2^{j-1}} \leq d^j 2^{j(d-j+1)} c^{(3^j-1)/2},$$

and so, for all $k \in [d-j]$,

$$\prod_{i=1}^k a_{j+i} > \frac{C_{j+k}}{d^j 2^{j(d-j+1)}} c^{(3^{j+k}-1)/2 - (3^j-1)/2} \geq \frac{C_{j+k}}{d^j 2^{j(d-j+1)}} c^{3^j(3^k-1)/2}.$$

Let $c' = d^2 2^{2j(d-j+1)} c^{3^j}$. (Note that $c' \geq c \cdot \prod_{i=1}^j a_i^2$.) Then, for all $k \in [d-j]$,

$$\begin{aligned} \prod_{i=1}^k a_{j+i} &> \frac{C_{j+k}}{d^j 2^{j(d-j+1)}} \left(\frac{c'}{d^2 2^{2j(d-j+1)}} \right)^{(3^k-1)/2} \\ &= \frac{(d2^d)^{\frac{3}{2}(3^{j+k-1}-1)}}{(d2^{d-j+1})^{j3^k}} c'^{(3^k-1)/2} \\ &\geq (d2^d)^{\frac{3}{2}(3^{j+k-1}-1)-j3^k} c'^{(3^k-1)/2} \\ &\geq (d2^d)^{\frac{3}{2}(3^{j+k-1}-1)-(3^j-1)3^k/2} c'^{(3^k-1)/2} \end{aligned}$$

because $t \leq (3^t - 1)/2$ for all $t \geq 1$. Continuing the computation,

$$\begin{aligned} \prod_{i=1}^k a_{j+i} &> (d2^d)^{\frac{3}{2}(3^{j+k-1}-1)-(3^j-1)3^k/2} c'^{(3^k-1)/2} \\ &= (d2^d)^{\frac{3}{2}(3^{j+k-1}-1-3^{j+k-1}+3^{k-1})} c'^{(3^k-1)/2} \\ &= (d2^d)^{\frac{3}{2}(3^{k-1}-1)} c'^{(3^k-1)/2} \\ &= C_k c'^{(3^k-1)/2}. \end{aligned}$$

Therefore $\bar{L}_j = [a_{j+1}, \dots, a_d]$ is c' -uncolorable by the inductive hypothesis. (It is easy to see that the C_j 's are increasing in d , so taking $d' = d - j$ causes no problem here.) Since L_j is also c -uncolorable by the inductive hypothesis, we may apply Lemma 10 to conclude that L is c -uncolorable.

5. Upper bounds on the volume of obstruction grids. Before proceeding, we introduce the following notation. For $d \geq 1$ and any monotone grid $L = [a_1, \dots, a_d]$, where $a_d > 1$, let $j \in [d]$ be least such that $a_j = a_d$. Then we let L^- denote the monotone grid obtained from L by subtracting one from a_j , that is,

$$L^- = [a_1, \dots, a_{j-1}, a_j - 1, a_{j+1}, \dots, a_d].$$

Note that if L is monotone and $L \in \mathcal{O}(c, d)$, then L is c -uncolorable but L^- is c -colorable. (Recall that $\mathcal{O}(c, d)$ is the obstruction set for c colors in d dimensions, i.e., the set of minimally c -uncolorable d -dimensional grids.)

The next theorem gives an asymptotic upper bound on the volume $\prod_{i=1}^d a_i$ of any grid $[a_1, \dots, a_d] \in \mathcal{O}(c, d)$.

THEOREM 11. *For every $d \geq 1$ and every grid $L = [a_1, \dots, a_d] \in \mathcal{O}(c, d)$,*

$$\prod_{i=1}^d a_i = O(c^{(3^d-1)/2})$$

as $c \rightarrow \infty$.

The theorem follows immediately from the following lemma.

LEMMA 12. *For every $d \geq 1$, every $c \geq 2$, and every monotone grid $L = [a_1, \dots, a_d] \in \mathcal{O}(c, d)$, there is a set $P \subseteq [d]$ such that*

- (i) $d \in P$,
- (ii) for every $\ell \in P$,

$$\prod_{i=1}^{\ell} a_i = O(c^{(3^{\ell}-1)/2})$$

as $c \rightarrow \infty$, and

- (iii) for every $k \in [d]$,

$$a_k = O(c^{3^j \cdot 2^{\ell-j-1}})$$

as $c \rightarrow \infty$, where ℓ is the least element of P that is $\geq k$ and j is the biggest element of P that is $< k$ ($j = 0$ if there is no such element).

(We call the elements of P *pinch points* for L .)

Proof. Let $d \geq 1$ and $c \geq 2$ be given, and let $L = [a_1, \dots, a_d] \in \mathcal{O}(c, d)$ be a monotone grid. Then L is c -uncolorable, and thus L_j is also c -uncolorable for all $1 \leq j \leq d$. Since $L \in \mathcal{O}(c, d)$, we have that L^- is c -colorable, and thus $\varepsilon_c(L^-) \geq 1$. This, in turn, implies that there is some largest $\ell \in [d]$ such that

$$2^{d-\ell} \frac{c^{2^{\ell-1}}}{a_{\ell} - 2} \geq \frac{1}{d}.$$

(Note that the denominator is positive because $a_{\ell} \geq a_1 \geq c + 1 \geq 3$ since L is c -uncolorable.) Thus,

$$(2) \quad a_{\ell} \leq d2^{d-\ell} \cdot c^{2^{\ell-1}} + 2 \leq (d+2)2^{d-\ell} \cdot c^{2^{\ell-1}},$$

and thus

$$(3) \quad \prod_{i=1}^{\ell} a_i \leq (a_{\ell})^{\ell} \leq ((d+2)2^{d-\ell})^{\ell} \cdot c^{\ell \cdot 2^{\ell-1}} \leq ((d+2)2^{d-1})^d \cdot c^{(3^{\ell}-1)/2},$$

which implies that ℓ satisfies (ii). We will make ℓ the least element of P , noticing that (2) and the monotonicity of L imply that a_k satisfies (iii) of the lemma for all $k \in [\ell]$ (with $j = 0$).

If $\ell = d$, then we let $P = \{\ell\} = \{d\}$, and we are done.

Otherwise, $\ell < d$. Note that $L^- = L_{\ell} \times (\bar{L}_{\ell})^-$ up to a possible permutation of the coordinates. Recall also that L_{ℓ} is not c -colorable, but L^- is. It follows from Lemma 10 that $(\bar{L}_{\ell})^-$ is c' -colorable, where

$$c' := c \cdot \prod_{i=1}^{\ell} \binom{a_i}{2} \leq c \cdot \left(\prod_{i=1}^{\ell} a_i \right)^2.$$

The bound in (3) gives $c' = O(c^{3^{\ell}})$ as $c \rightarrow \infty$.

We thus have $\varepsilon_{c'}((\bar{L}_{\ell})^-) \geq 1$, and so there is some largest m with $\ell < m \leq d$ such that

$$2^{d-m} \frac{(c')^{2^{m-\ell-1}}}{a_m - 2} \geq \frac{1}{d - \ell},$$

which gives

$$(4) \quad a_m \leq (d - \ell)2^{d-m} \cdot (c')^{2^{m-\ell-1}} + 2$$

$$(5) \quad \leq (d - \ell + 2)2^{d-m} \cdot (c')^{2^{m-\ell-1}}$$

$$(6) \quad = O(c^{3^{\ell} \cdot 2^{m-\ell-1}})$$

as $c \rightarrow \infty$. For the volume of L_m , we get

$$\begin{aligned} \prod_{i=1}^m a_i &= \prod_{i=1}^{\ell} a_i \cdot \prod_{i=\ell+1}^m a_i \\ &\leq \left(\prod_{i=1}^{\ell} a_i \right) \cdot (a_m)^{m-\ell} \\ &= O(c^{(3^{\ell}-1)/2}) \cdot O(c^{3^{\ell} \cdot (m-\ell) \cdot 2^{m-\ell-1}}) \\ &= O(c^{(3^{\ell}-1)/2} \cdot c^{3^{\ell} \cdot (3^{m-\ell}-1)/2}) \\ &= O(c^{(3^m-1)/2}) \end{aligned}$$

as $c \rightarrow \infty$. We make ℓ and m the two least elements of P , and the last calculation shows that $m \in P$ satisfies (ii). Further, since $a_k \leq a_m$ for all k such that $\ell < k \leq m$, (iii) is also satisfied for all these a_k by (4)–(6).

If $m = d$, then we let $P = \{\ell, m\}$, and we are done. Otherwise, we repeat the argument above using m instead of ℓ to obtain an n with $m < n \leq d$ such that ℓ, m , and n

being the least three elements of P satisfies (ii) and (iii) and so on until we arrive at d , whence we set $P := \{\ell, m, n, \dots, d\}$.

The next proposition shows that the bounds in Lemma 12 are asymptotically tight.

PROPOSITION 13. *For $c \geq 2$, there is an infinite sequence $\{\mu_j(c)\}_{j=1}^{\infty}$ of positive integers such that*

- (i) $\mu_j(c) \geq 1 + 2^{(1-3^{j-1})/2} \cdot c^{3^{j-1}}$ for all $j \in \mathbb{Z}^+$ and
- (ii) for all $d \geq 1$, the grid $[\mu_1(c), \dots, \mu_d(c)] \in \mathcal{O}(c, d)$ with pinch point set $P = [d]$.

Proof. For all $c \geq 2$, define

$$\begin{aligned} \mu_1(c) &:= 1 + c, \\ \mu_2(c) &:= 1 + c \cdot \binom{c+1}{2}, \\ &\vdots \\ \mu_{j+1}(c) &:= 1 + c \cdot \prod_{i=1}^j \binom{\mu_i(c)}{2}, \\ &\vdots \end{aligned}$$

Fix $c \geq 2$, and let μ_j denote $\mu_j(c)$ for short. A routine induction on j shows (i). For the inductive step, noting that $\sum_{i=0}^{j-1} 3^i = (3^j - 1)/2$, we have

$$\begin{aligned} \mu_{j+1} &= 1 + c \cdot \prod_{i=1}^j \binom{\mu_i}{2} \\ &\geq 1 + \frac{c}{2^j} \prod_{i=1}^j (\mu_i - 1)^2 \\ &\geq 1 + \frac{c}{2^j} \prod_{i=1}^j \frac{c^{2 \cdot 3^{i-1}}}{2^{3^{i-1}-1}} \\ &= 1 + \frac{c^{3^j}}{2^{(3^j-1)/2}}. \end{aligned}$$

For (ii), we use induction on $d \geq 1$ to show separately that

- (a) $[\mu_1, \dots, \mu_d]$ is c -uncolorable and
- (b) $[\mu_1, \dots, \mu_d]$ is not $(c, 2)$ -guaranteed (i.e., there is a coloring $[\mu_1, \dots, \mu_d] \rightarrow [c]$ that monocolored exactly one box).

Clearly $[\mu_1] = [1 + c]$ is c -uncolorable by the pigeonhole principle. Now let $d \geq 2$, and assume that $[\mu_1, \dots, \mu_{d-1}]$ is c -uncolorable. Then letting $c' = c \cdot \prod_{i=1}^{d-1} \binom{\mu_i}{2}$, we have $\mu_d = 1 + c'$, and hence $[\mu_d]$ is c' -uncolorable. But then $[\mu_1, \dots, \mu_d]$ is c -uncolorable by Lemma 10 (letting $j = d - 1$).

We now show (b). For $d = 1$, clearly the coloring $[\mu_1] \rightarrow [c]$ mapping $j \mapsto (j \bmod c) + 1$ has exactly one monochromatic one-dimensional box, namely, $(1; c) = \{1, c + 1\}$. Now let $d \geq 2$, and assume (b) holds for $d - 1$, i.e., there is a coloring $[\mu_1, \dots, \mu_{d-1}] \rightarrow [c]$ that monocolored exactly one box. We will call such a coloring *minimal*. This generates exactly $\prod_{i=1}^{d-1} \binom{\mu_i}{2}$ many boxes in $[\mu_1, \dots, \mu_{d-1}]$. For each of these boxes B and for each color s , we can find a minimal coloring that monocolored B with s by

permuting the order of the hyperplanes along each axis and by permuting the colors. Thus there are exactly $c' = c \cdot \prod_{i=1}^{d-1} \binom{\mu_i}{2}$ many distinct minimal colorings. We overlay these c' many colorings to obtain a coloring of $[\mu_1, \dots, \mu_{d-1}, c']$ with no monochromatic d -boxes. We then duplicate the first $(d-1)$ -dimensional layer to arrive at a c -coloring of $[\mu_1, \dots, \mu_{d-1}, 1 + c'] = [\mu_1, \dots, \mu_d]$. This coloring has only one monocolored d -box: the box corresponding to the duplicated layer of unique monocolored $(d-1)$ -boxes. This shows (b).

It follows from (b) that $[\mu_1, \dots, \mu_{d-1}, \mu_d - 1]$ is c -colorable for all $d \geq 1$ since we can remove a single hyperplane from the only monocolored d -box in some minimal coloring of $[\mu_1, \dots, \mu_{d-1}, \mu_d]$ to leave a coloring of $[\mu_1, \dots, \mu_{d-1}, \mu_d - 1]$ without any monochromatic $(d-1)$ -boxes. From this it easily follows that $[\mu_1, \dots, \mu_d] \in \mathcal{O}(c, d)$ because $[\mu_1, \dots, \mu_{j-1}, \mu_j - 1]$ is c -colorable, and hence $[\mu_1, \dots, \mu_{j-1}, \mu_j - 1, \mu_{j+1}, \dots, \mu_d]$ is c -colorable for all $j \in [d]$.

Finally it is evident that all $j \in [d]$ are pinch points for $[\mu_1, \dots, \mu_d]$. (It is interesting to note that $[\mu_1, \dots, \mu_d]$ is the lexicographically first element of $\mathcal{O}(c, d)$.)

6. Upper bound on the size of the obstruction set. It was shown in [7] that $|\mathcal{O}(c, 2)| \leq 2c^2$. We give an asymptotic upper bound for $|\mathcal{O}(c, d)|$ for every fixed $d \geq 3$.

THEOREM 14. *For all $d \geq 3$,*

$$|\mathcal{O}(c, d)| = O(c^{(17 \cdot 3^{d-3} - 1)/2})$$

as $c \rightarrow \infty$.

Proof. Fix $d \geq 3$. We give an asymptotic upper bound on the number of monotone grids in $\mathcal{O}(c, d)$. The size of $\mathcal{O}(c, d)$ is at most $d!$ times this bound, and so it is asymptotically equivalent. By Lemma 12, every grid $L \in \mathcal{O}(c, d)$ has a set P of pinch points. For each set $P \subseteq [d]$ such that $d \in P$, let $\#_c(P)$ be the number of monotone grids in $\mathcal{O}(c, d)$ having pinch point set P . There are 2^{d-1} many such P , so an asymptotic bound on $\max\{\#_c(P) | P \subseteq [d] \wedge d \in P\}$ gives the same asymptotic bound on $|\mathcal{O}(c, d)|$.

Fix a set $P \subseteq [d]$ such that $d \in P$, and let $P = \{\ell_1 < \ell_2 < \dots < \ell_s = d\}$, where $s = |P|$ and ℓ_1, \dots, ℓ_s are the elements of P in increasing order. For convenience, set $\ell_0 := 0$. Lemma 12 says that, for any monotone grid $L = [a_1, \dots, a_d] \in \mathcal{O}(c, d)$ having pinch point set P , for any $b \in [s]$, and for any k such that $\ell_{b-1} < k \leq \ell_b$, we have $a_k = O(c^{e(b)})$ as $c \rightarrow \infty$, where

$$e(b) := 3^{\ell_b - 1} \cdot 2^{\ell_b - \ell_{b-1} - 1}.$$

To bound $\#_c(P)$, we first note that, for any choice of $1 \leq a_1 \leq \dots \leq a_{d-1}$, there can be at most one value of a_d such that $[a_1, \dots, a_d] \in \mathcal{O}(c, d)$ because any two d -dimensional grids that share the first $d-1$ dimensions are comparable in the dominance order \leq . Thus $\#_c(P)$ is bounded by the number of possible combinations of values of a_1, \dots, a_{d-1} . From the bound on each a_k given by Lemma 12(iii), we therefore have

$$\begin{aligned} \#_c(P) &\leq \left(\prod_{b=1}^{s-1} \prod_{k=\ell_{b-1}+1}^{\ell_b} O(c^{e(b)}) \right) \cdot \prod_{k=\ell_{s-1}+1}^{d-1} O(c^{e(s)}) \\ &= O\left(\prod_{b=1}^{s-1} (c^{e(b)})^{\ell_b - \ell_{b-1}} \right) \cdot O((c^{e(s)})^{d-1 - \ell_{s-1}}) \\ &= O(c^{h_1 + h_2}) \end{aligned}$$

as $c \rightarrow \infty$, where $h_2 = e(s)(d-1 - \ell_{s-1})$ and

$$\begin{aligned}
h_1 &= \sum_{b=1}^{s-1} e(b)(\ell_b - \ell_{b-1}) \\
&= \sum_{b=1}^{s-1} 3^{\ell_{b-1}} \cdot 2^{\ell_b - \ell_{b-1} - 1} \cdot (\ell_b - \ell_{b-1}) \\
&\leq \sum_{b=1}^{s-1} 3^{\ell_{b-1}} \cdot \frac{3^{\ell_b - \ell_{b-1}} - 1}{2} \\
&= \frac{1}{2} \sum_{b=1}^{s-1} (3^{\ell_b} - 3^{\ell_{b-1}}) \\
&= \frac{3^m - 1}{2},
\end{aligned}$$

where $m = \ell_{s-1}$. We also have

$$\begin{aligned}
h_2 &= 3^{\ell_{s-1}} \cdot 2^{d - \ell_{s-1} - 1} \cdot (d - 1 - \ell_{s-1}) \\
&= 3^m \cdot 2^{d-m-1} \cdot (d - m - 1),
\end{aligned}$$

whence

$$h_1 + h_2 = \frac{3^m - 1}{2} + 3^m \cdot 2^{d-m-1} \cdot (d - m - 1).$$

This expression depends on only the value of m , which satisfies $0 \leq m < d$. It is more convenient to express $h_1 + h_2$ in terms of $n := d - m$, where $n \in [d]$:

$$h_1 + h_2 = \frac{3^{d-n} - 1}{2} + 3^{d-n} \cdot 2^{n-1} \cdot (n - 1) = \frac{3^d}{2} \cdot \frac{1 + 2^n(n - 1)}{3^n} - \frac{1}{2}.$$

It is easy to check that $(1 + 2^n(n - 1))/3^n$ is greatest (and thus $h_1 + h_2$ is greatest) when $n = 3$. It follows that

$$h_1 + h_2 \leq \frac{3^d}{2} \cdot \frac{1 + 2^3(3 - 1)}{3^3} - \frac{1}{2} = \frac{17 \cdot 3^{d-3} - 1}{2},$$

which proves the theorem.

The first few values $(17 \cdot 3^{d-3} - 1)/2$ are given in Table 1.

7. Three dimensions and two colors. In this section, we extend to three dimensions the result of [7, Theorem 6.2], which states that $\mathcal{O}(2, 2) = \{[7, 3], [5, 5], [3, 7]\}$. The

TABLE 1
Table of upper bounds on e so that $|\mathcal{O}(c, d)| = O(c^e)$ as $c \rightarrow \infty$.

d	$(17 \cdot 3^{d-3} - 1)/2$
3	8
4	25
5	76
6	229

following graph (Figure 1, generated using the Jmol module in SAGE) and table (Table 2) display upper bounds for the smallest a_3 so that $[a_1, a_2, a_3]$ is 2-uncolorable. All three graphical axes run from 3 to 130; the table includes only $3 \leq a_1 \leq 12$ and $3 \leq a_2 \leq 12$. We believe these values to be very close to the truth; indeed, we have matching lower bounds in many cases and lower bounds that differ from the upper bounds by at most 2 in many more cases.

A few different methods were applied to obtain these bounds. First the values Δ_j , as in section 3, were computed, and the least a_3 so that $\Delta_3 > 0$ was recorded. In fact, this

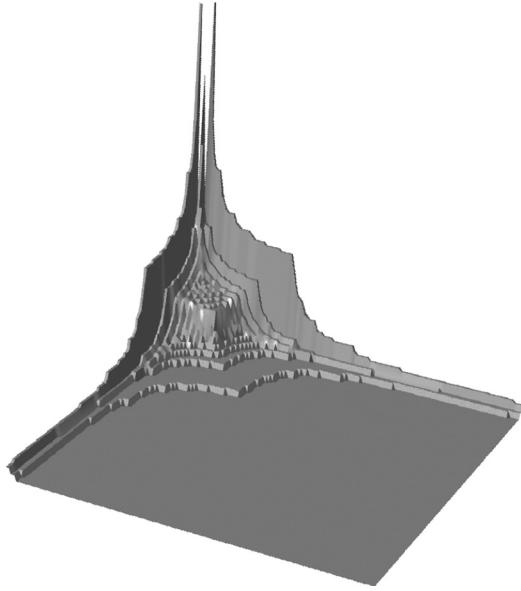


FIG. 1. *Graph of upper bounds on a_3 so that $[a_1, a_2, a_3]$ 2-uncolorable.*

TABLE 2
Table of bounds on a_3 so that $[a_1, a_2, a_3]$ is 2-uncolorable.

	3	4	5	6	7	8	9	10	11	12
3					127	85	73	68	67	67
4					127	85	73	68	67	67
5			101	76	53	47	46	46	40	37
6			76	76	53	47	46	46	40	37
7	127	127	53	53	53	46	40	37	34	33
8	85	85	47	47	46	45	40	37	34	33
9	73	73	46	46	40	40	37	34	31	30
10	68	68	46	46	37	37	34	33	31	30
11	67	67	40	40	34	34	31	31	30	28
12	67	67	37	37	33	33	30	30	28	28

idea was improved slightly by applying the observation that, if some grid is $(2, t)$ -guaranteed, then it is $(2, \lceil t \rceil)$ -guaranteed. In some cases, this increases the value of Δ_j . Second we used the simple observations that c -colorability is independent of the order of the a_i and that $L \preccurlyeq L'$ when L is c -uncolorable implies that L' is c -uncolorable. Third we applied the following lemma.

LEMMA 15. *Let $M = \prod_{j=1}^d \binom{a_j}{2}$. If the grid $L = [a_1, \dots, a_d]$ is (c, t) -guaranteed, then $L \times [\lfloor cM/t \rfloor + 1]$ is c -uncolorable.*

Proof. Note that $K = \lfloor cM/t \rfloor + 1 > cM/t$ and is integral. If we think of $L \times [K]$ as K copies of L , then any c -coloring of $L \times [K]$ restricts to K c -colorings of L . Since L is (c, t) -guaranteed, each of these c -colorings gives rise to t monochromatic boxes. Hence, in K colorings, there are at least $t(\lfloor cM/t \rfloor + 1) > cM$ monochromatic boxes. Since there are only M total boxes in each copy of L and any monochromatic box can be colored only in c different ways, there must be two identical boxes (in two different copies of L) which are monochromatic and have the same color. This is precisely a monochromatic $(d+1)$ -dimensional box in $L \times [K]$.

Therefore, in order to obtain upper bounds on $[a_3]$ in the above table, we need to know the greatest t for which $[a_1] \times [a_2]$ is $(2, t)$ -guaranteed. To that end, we define the following matrix.

DEFINITION 16. *Define $f_j: [r] \rightarrow [2]$, $0 \leq j < 2^r$, to be the function which maps $i \in [r]$ to the coefficient of 2^{i-1} in the base-2 expansion of j . Define M_r to be the $2^r \times 2^r$ integer matrix whose (i, j) -entry is given by*

$$\binom{|f_i^{-1}(1) \cap f_j^{-1}(1)|}{2} + \binom{|f_i^{-1}(2) \cap f_j^{-1}(2)|}{2}.$$

Then define the quadratic form $Q_r: \mathbb{R}^{2^r} \rightarrow \mathbb{R}$ by $Q_r(\mathbf{v}) = \mathbf{v}^ M_r \mathbf{v}$. Let*

$$\delta_r = (M_r(1, 1), \dots, M_r(2^r, 2^r))$$

be the diagonal of M_r , and define $\mathbf{1}$ to be the all-ones vector of length 2^r .

PROPOSITION 17. *Let t be the least value of $(Q_r(\mathbf{v}) - \mathbf{v}^* \delta_r)/2$ over all nonnegative integer vectors $\mathbf{v} \in \mathbb{Z}^{2^r}$ with $\mathbf{v}^* \mathbf{1} = s$. Then $[r] \times [s]$ is (c, t) -guaranteed, and t is the maximum value so that this is the case.*

Proof. Given a vector $\mathbf{v} = (v_1, \dots, v_r)$ satisfying the hypotheses, consider the $r \times s$ matrix A with v_j many columns of type f_j for each $j \in [r]$. (We may identify f_j with a column vector in $[2]^r$ in the natural way.) It is easy to see that $Q_r(\mathbf{v}) - \mathbf{v}^* \delta_r$ exactly counts twice the number of monochromatic rectangles in A , thought of as a 2-coloring of the grid $[r] \times [s]$.

We applied standard quadratic integer programming tools (XPress-MP) to minimize the appropriate programs. Fortunately, for the cases considered, the matrix M_r was positive semidefinite, meaning that the solver could use polynomial time convex programming techniques during the interior point search. We conjecture that this is always the case.

CONJECTURE 18. *M_r is positive semidefinite for $r \geq 3$.*

In particular, for $r = 3$, the eigenvalues of M_r are 0, 1, and 4, with multiplicities 2, 4, and 2, respectively. For $4 \leq r \leq 9$, the eigenvalues are 0, 2^{r-2} , $2^{r-3}(r-2)$, $2^{r-2}(r-1)$, and $2^{r-4}(r^2 - r + 2)$, with multiplicities $2^r - r(r+1)/2$, $r(r-1)/2 - 1$, $r-1$, 1, and 1, respectively. We conjecture that this description of the spectrum is valid for all $r \geq 4$.

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