Determinant & trace

Fact: Let $m, n \geq 1$. Let
A be $m \times n$ matrix
B be $n \times m$ matrix
Then $tr(AB) = tr(BA)$

Notation: A matrix, $i, j \geq 1$.
$[A]_{ij}$ for the $(i,j)^{th}$ entry of A

$$tr(AB) = \sum_{j=1}^{m} [AB]_{jj}$$
$$= \sum_{j=1}^{m} \sum_{k=1}^{n} [A]_{jk} [B]_{kj}$$
$$= \sum_{k=1}^{n} \sum_{j=1}^{m} [B]_{kj} [A]_{jk}$$
$$= \sum_{k=1}^{n} [BA]_{kk} = tr(BA)$$

Last time: A is square matrix
A normal $\iff AA^* = A^*A$
A Hermitian $\iff A = A^*$
A unitary $\iff AA^* = I$ (identity)
$(\iff A^*A = I)$

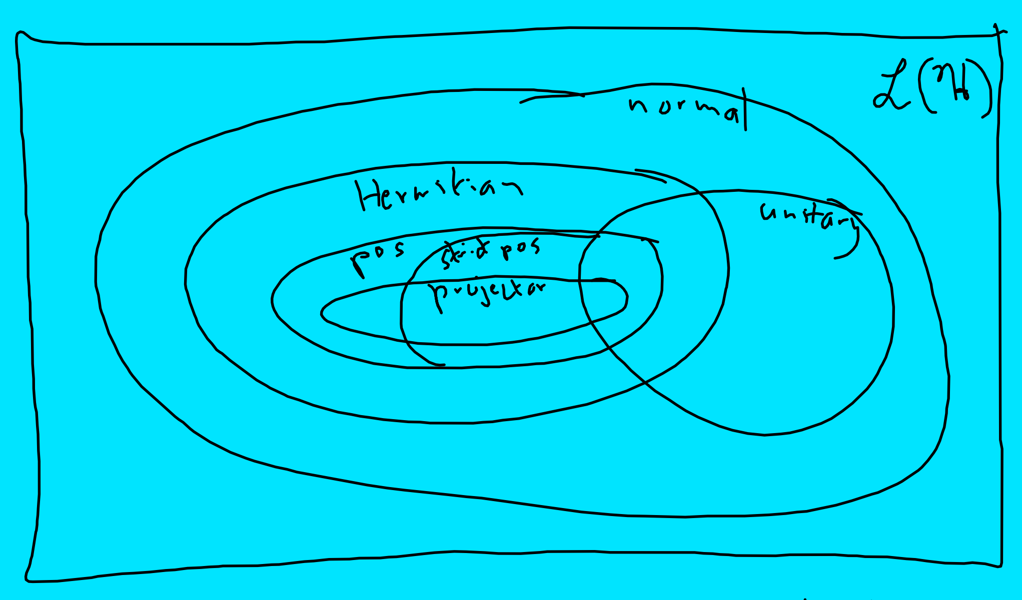
All vector spaces we mention will be finite-dimensional.

$\mathbb{C}$-space $\equiv$ finite dimensional Hilbert space

Def:
A is positive (positive semidefinite) $(A \geq 0)$ if $A = B^* B$
for some B.
Equiv: $v^* A v \geq 0$ for all vectors v

A is strictly positive (positive definite) $(A > 0)$ if $A \geq 0$ and invertible.

A is a projector (orthogonal projector)
if $AA = A$ (A is idempotent)
and $A = A^*$ (A is Hermitian)



Unitary operators. Let U be unitary. Let v, w be vectors.

$$\langle Uv, Uw \rangle = \langle U^* U v, w \rangle$$
$$= \langle I v, w \rangle = \langle v, w \rangle$$

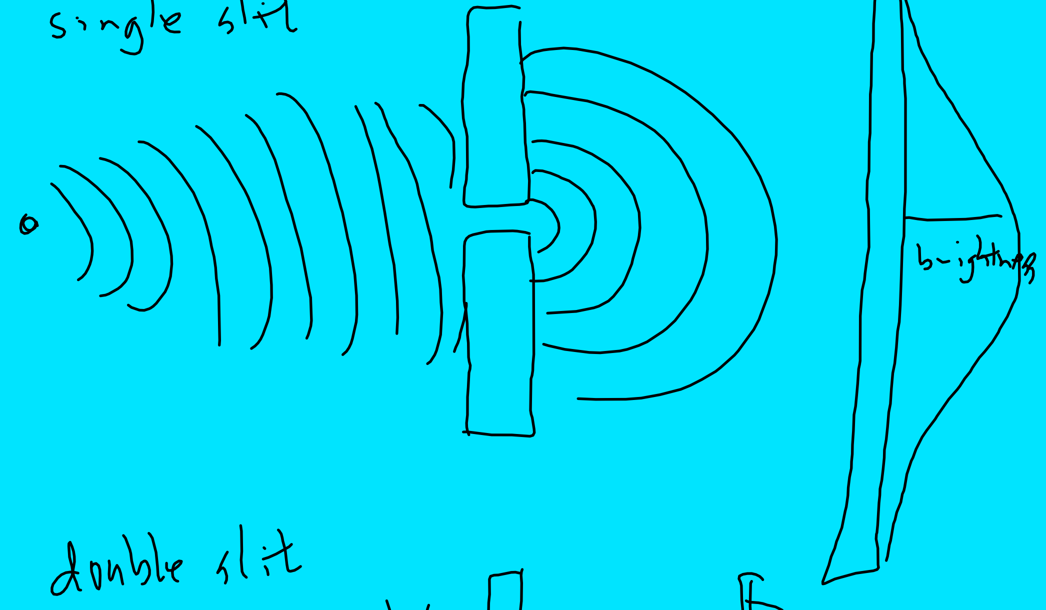
U preserves inner products

Particularly, $\{Ue_1, Ue_2, \ldots, Ue_n\}$ is an orthonormal basis.
[Any orth. basis can be obtained this way: applying a unitary op to the standard basis.]

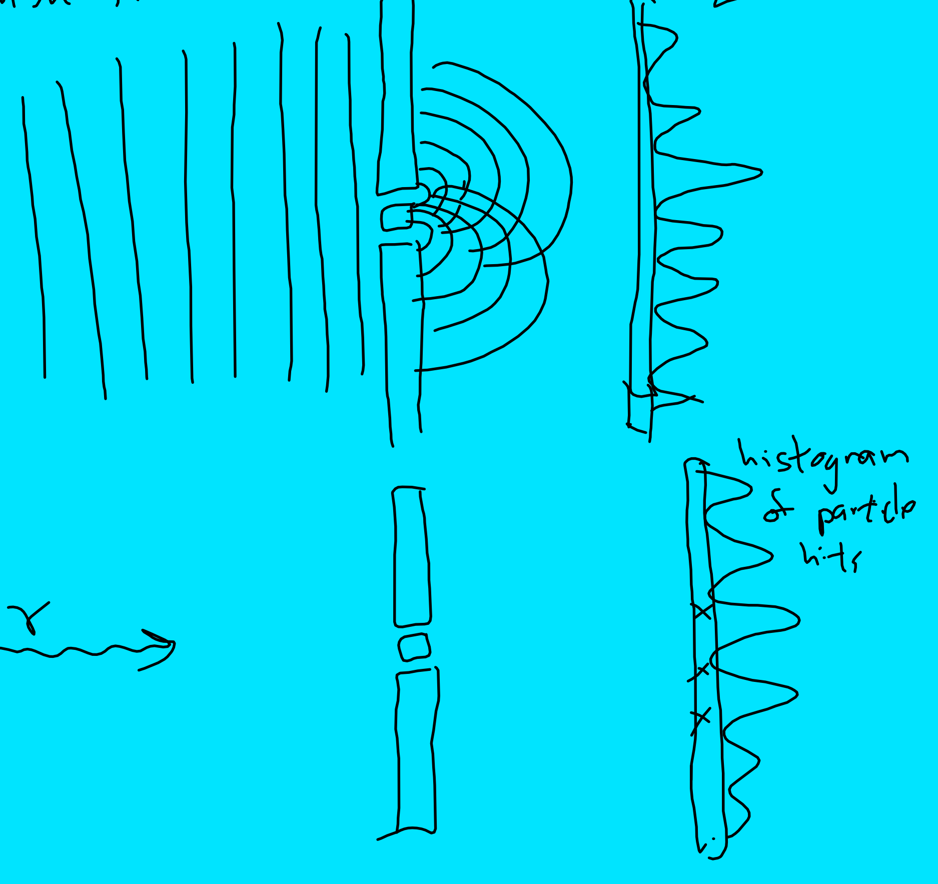
Some physics

Double slit experiment
Stern-Gerlach experiment

single slit

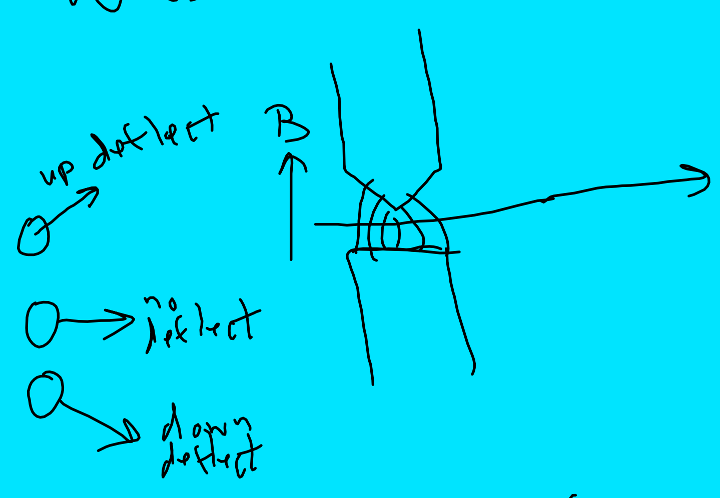


double slit



Stern-Gerlach experiment

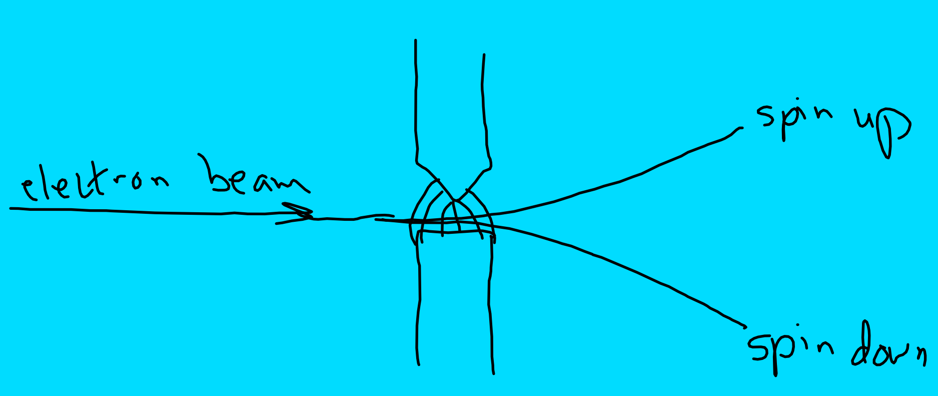
Classical physics

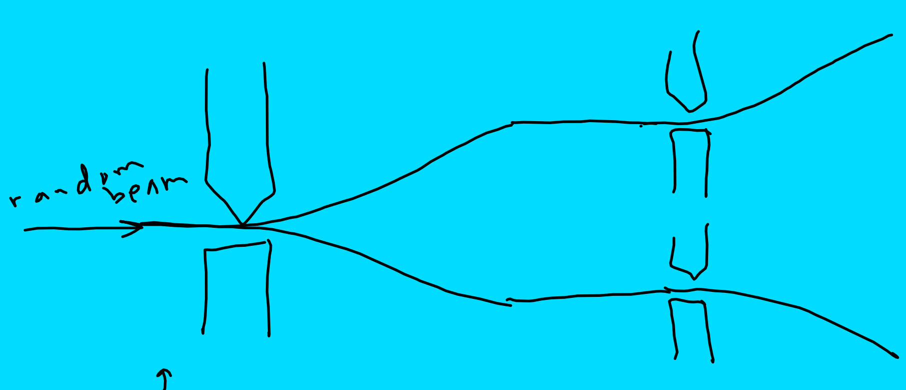


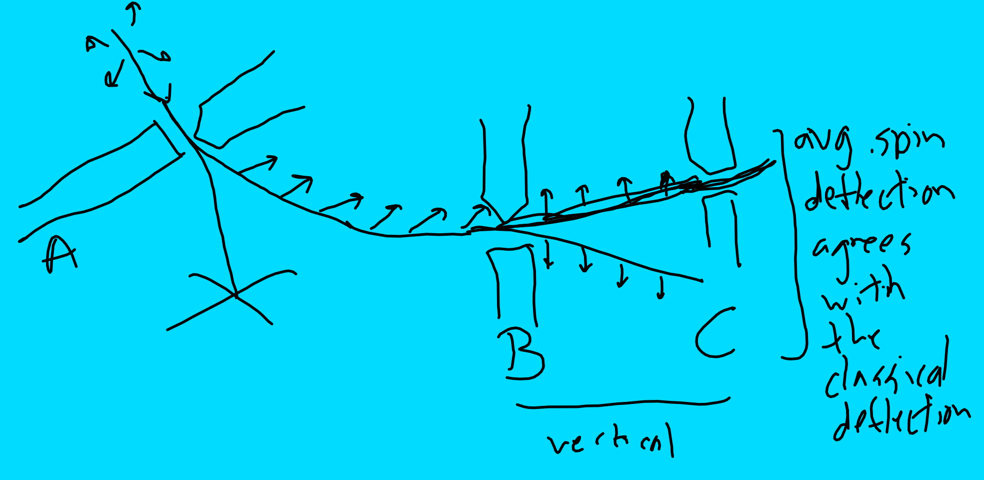
deflection $\propto$ (spin vector) $\cdot$ (field gradient)

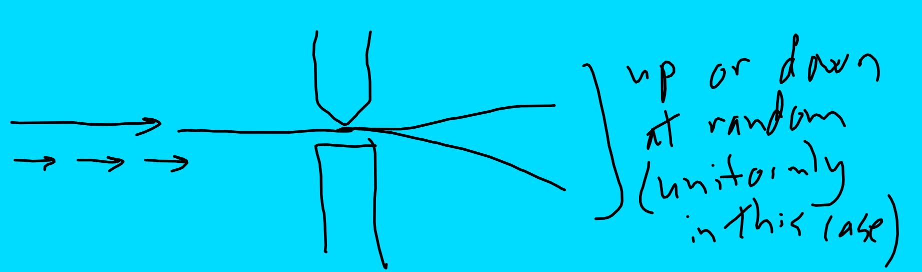
proportional to

dot product









(1) Nature is inherently random

(2) wave-like interference occurs, which affects the statistics.

QM explains both — predicts the statistical distribution based on physical forces.

Quantum Bit (qubit)

This is a quantum system with two possible values: 0 or 1

(quantum version of a bit)

A (bounded) physical system corresponds to a $\mathbb{C}$-space $\mathcal{H}$

the dimension of $\mathcal{H}$ is the max number of possible values observed on the system.

Qubit — 2-dimensional $\mathcal{H}$.

E.g., electron spin (spin up, spin down)

Pick an orth. basis for $\mathcal{H}$:

$$\{|0\rangle, |1\rangle\} \text{ computational basis}$$

A state of a system is a unit vector in the $\mathbb{C}$-space (for now)

General 1-qubit state is of the form

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

where $\alpha, \beta \in \mathbb{C}$.

This is a unit vector, so

$$1 = \langle |\psi\rangle, |\psi\rangle \rangle = (|\psi\rangle)^* |\psi\rangle$$
