Addition, multiplication of complex conjugates:

$$z = x + iy \qquad (x, y \in \mathbb{R})$$

$$\bar{z} = z^* = x - iy$$

$$w^* + z^* = (w+z)^*$$

$$(wz)^* = w^* z^*$$

$$|z| = \sqrt{z^* z} = \sqrt{x^2 + y^2}$$

Exp map $\exp : \mathbb{C} \to \mathbb{C}$

$$\exp(z) = e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!}$$

$$\exp(z+w) = \exp(z)\exp(w)$$

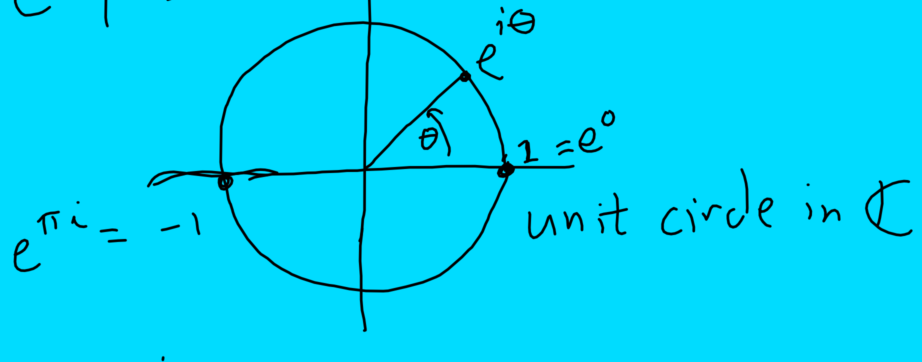
$$e^0 = 1$$

$$e^{-z} = 1/e^z$$

Euler's fmla: $\theta \in \mathbb{R}$

$$e^{i\theta} = \cos\theta + i \sin\theta$$

$$|e^{i\theta}| = 1$$



$$e^{2\pi i} = 1 = e^0$$

exp is periodic with period $2\pi i$

$$e^{\pi i} = -1 \qquad \boxed{e^{\pi i} + 1 = 0}$$

$\mathbb{C}^n$ is the vector space of n-dim column vectors (over $\mathbb{C}$) $(n \geq 1)$

<u>Standard basis</u>: $\{e_1, e_2, \ldots, e_n\}$

$$\begin{array}{cl} e_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow \text{j'th position} \\ \\ \\ \\ \end{array} \end{array}$$

vector $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \sum_{j=1}^{n} v_j e_j$

$v_1, \ldots, v_n \in \mathbb{C}$

Matrices $m, n \geq 1$.

$\mathbb{C}^{m \times n}$ is the vector space (over $\mathbb{C}$) of all $m \times n$ matrices

m rows
n columns

$A \in \mathbb{C}^{m \times n}$. view A as a linear map $: \mathbb{C}^n \to \mathbb{C}^m$

$$v \mapsto Av \quad (v \in \mathbb{C}^n)$$

<u>Adjoints</u>: $A \in \mathbb{C}^{m \times n}$

The <u>adjoint</u> (Hermitian conjugate) of A is the $n \times m$ matrix A^* $(A^\dagger)$ which is the <u>conjugate transpose</u> of A: take the cpx conjugate of each entry of A and then take the transpose.

<u>Ex</u>: $A = \begin{bmatrix} -3 & 2+i & 6-3i \\ -i & 1 & -2-7i \end{bmatrix}$

$$A^* = \begin{bmatrix} -3 & i \\ 2-i & 1 \\ 6+3i & -2+7i \end{bmatrix}$$

Properties:

$$A^{**} = A \qquad \begin{pmatrix} z \in \mathbb{C} \\ z^{**} = z \end{pmatrix}$$

$$(A+B)^* = A^* + B^*$$

$$\left[A \in \mathbb{R}^{m \times n} \Rightarrow A^* = A^T \right]$$

$$(AB)^* = B^* A^*$$

Hermitian Innerproduct on $\mathbb{C}^n$

Given $v, w \in \mathbb{C}^n$,

define $\langle v, w \rangle := v^* w \in \mathbb{C}$

$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, $w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$ $v_j, w_j \in \mathbb{C}$,

$$\underset{v^*w}{\langle v, w \rangle} = \begin{bmatrix} v_1^* & \cdots & v_n^* \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

$$= v_1^* w_1 + v_2^* w_2 + \cdots + v_n^* w_n$$

$$= \sum_{j=1}^{n} v_j^* w_j .$$

Note: $\langle v, v \rangle = \sum_{j=1}^{n} v_j^* v_j = \sum_{j=1}^{n} |v_j|^2$

$\underbrace{\geq 0}_{\text{assume real}}$ and $= 0$ iff $v = 0$

[lots of stuff lost]

Property (crucial) of the adjoint: $u \in \mathbb{C}^m$, $v \in \mathbb{C}^n$, $A = \mathbb{C}^{m \times n}$.

$$\langle u, Av \rangle = \langle A^* u, v \rangle.$$

<u>Pf</u>:

$$\langle u, Av \rangle = u^* A v = (A^* u)^* v$$
$$= \langle A^* u, v \rangle$$

A <u>$\mathbb{C}$-space</u> is a vector space $\mathbb{C}^n$ with the Hermitian inner product.

$\mathcal{H}, \mathcal{J}$ $\mathbb{C}$-spaces

$$\mathcal{H} = \mathbb{C}^m$$
$$\mathcal{J} = \mathbb{C}^n$$

Then $\mathcal{L}(\mathcal{H}, \mathcal{J}) = \mathbb{C}^{n \times m}$

space of linear operators ($n \times m$ matrices)

$\mathcal{H} \longrightarrow \mathcal{J}$.

If $\mathcal{H} = \mathcal{J}$, then write

$$\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$$

<u>linear operators on $\mathcal{H}$</u>;

<u>square operators</u> (square matrices)

<u>Def</u>: $A \in \mathbb{C}^{n \times n}$ be a square matrix.

A is <u>normal</u> if $AA^* = A^*A$

A is <u>Hermitian</u> if $A = A^*$

A is <u>unitary</u> if $A^*A = I$
$(= AA^*)$
(equiv, $A^* = A^{-1}$)