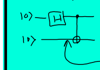
Evolving stabilizing group generators through a Clifford circuit by example.



stab generators

Initial $|00\rangle$ $\{Z_1, Z_2\}$

$= \{Z\otimes I, I\otimes Z\}$

$\xmapsto{H_1} \{\overset{H_1 Z_1 H_1}{\overset{\shortparallel}{X_1}}, Z_2\}$

$\xmapsto{C\text{-}X_{1,2}} \{\underset{X\otimes X}{X_1X_2}, \underset{Z\otimes Z}{Z_1Z_2}\}$

$G = \{1, X_1X_2, Z_1Z_2, -Y_1Y_2\}$

$X_1\underbrace{X_2Z_1}Z_2 = X_1Z_1X_2Z_2 = (-iY_1)(-iY_2)$

$= -Y_1Y_2$

Check matrix

$$M_{initial} = \begin{array}{c} Z_1 \\ Z_2 \end{array}\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

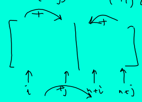
Clifford gate conjugation corresponds to column ops on the matrix:

H_j : swap j^{th} col with the $(n+j)^{th}$ col

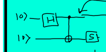
$$\xmapsto{H_1} M_1 = \begin{array}{c} X_1 \\ Z_2 \end{array}\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$C\text{-}X_{i,j}$: add i'th col to jth col
add $(n+j)^{th}$ col to $(n+i)^{th}$ col

$i \neq j$
i control
j target



$$\xmapsto{C\text{-}X_{1,2}} \begin{array}{c} X_1X_2 \\ Z_1Z_2 \end{array}\left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$



$\{X_1X_2, Z_1Z_2\}$

$\xmapsto{S_2} \{X_1Y_2, Z_1Z_2\}$

$SXS^* = Y$
$SYS^* = -X$
$SZS^* = Z$

S_j : add the j'th col to the $(n+j)^{th}$ col

$$M_{final} = \begin{array}{c} X_1Y_2 \\ Z_1Z_2 \end{array}\left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

when is a minus size generated

Applying S_2 one more time:

gives $S_2X_1Y_2S_2^* = X_1S_2Y_2S_2^*$

$= X_1(-X_2)$

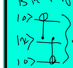
$= -X_1X_2$

$$(-)\left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

only other case a minus sign arizes:
$HYH = -Y$

Bit flip code encoding circuit:

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$



$\alpha|000\rangle + \beta|111\rangle$

Initially:

$\{Z_1, Z_3\}$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\xmapsto{C\text{-}X_{2,1}} \{Z_1Z_2, Z_3\}$ $\begin{array}{c} Z_1Z_2 \\ Z_3 \end{array}\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$

$\xmapsto{C\text{-}X_{2,3}} \{Z_1Z_2, Z_2Z_3\}$ $\begin{array}{c} Z_1Z_2 \\ Z_2Z_3 \end{array}\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$

Bit-flip code space is stabilized by $\langle \underline{Z_1Z_2}, \underline{Z_2Z_3} \rangle$

$= \{1, Z_1Z_2, Z_2Z_3, Z_1Z_3\}$

Shor code stab group generators

				5	6	7	8	9
Z	Z							
	Z	Z						
			Z	Z				
				Z	Z			
						Z	Z	
							Z	Z
X	X	X	X	X	X			
			X	X	X	X	X	X

rows are generators (blank entry $= I$)

Quantifying entanglement

Thm: (Schmidt decomposition)

$\mathcal{H}, \mathcal{J}$ be $\mathbb{C}$-spaces

$|\psi\rangle \in \mathcal{H}\otimes\mathcal{J}$ (unit vector)

There exist orthonormal sets

$\{|u_1\rangle, \ldots, |u_r\rangle\} \subseteq \mathcal{H}$

$\{|v_1\rangle, \ldots, |v_r\rangle\} \subseteq \mathcal{J}$

and unique $\lambda_1, \ldots, \lambda_r > 0$

such that $\sum_{j=1}^{r} \lambda_j = 1$

and $|\psi\rangle = \sum_{j=1} \sqrt{\lambda_j}\, |u_r\rangle \otimes |v_r\rangle$

$\lambda_1, \dots, \lambda_r$ are <u>Schmidt coefficients</u>

& r = <u>Schmidt number</u>.

$|\psi\rangle$ is entangled iff $r > 1$

$\left(r = 1 \Rightarrow \underbrace{|\psi\rangle = |u_1\rangle \otimes |v_1\rangle}_{\text{separable}} \right)$

$(\lambda_1, \dots, \lambda_r)$ determines the amount of entanglement, which is $H(\lambda_1, \dots, \lambda_r)$ (Shannon entropy)

$$:= \sum_{j=1}^{r} \lambda_j \log_2\left(\frac{1}{\lambda_j}\right)$$

$$= -\sum_j \lambda_j \log_2 \lambda_j$$

[convention: $0 \log_2 0 = 0$)

$$\underbrace{H\left(\frac{1}{r}, \dots, \frac{1}{r}\right)}_{\text{uniform dist}} = \log_2 r$$

$$0 \le H(\lambda_1, \dots, \lambda_r) \le \log_2 r$$

Start with $\rho = |\psi\rangle\langle\psi|$

$$\rho = \sum_{i,j} \lambda_i \lambda_j \underbrace{|u_i\rangle\langle u_j|}_{\in \mathcal{L}(\mathcal{H})} \otimes \underbrace{|v_i\rangle\langle v_j|}_{\in \mathcal{L}(\mathcal{K})}$$

$$\mathrm{Tr}_{\mathcal{K}}\, \rho = \sum_{i,j} \sqrt{\lambda_i \lambda_j}\, |u_i\rangle\langle u_j|\, \delta_{ij}$$

$$= \sum_j \underbrace{\lambda_j}_{\text{eigenvals of } \mathrm{Tr}_{\mathcal{K}}\rho} |u_j\rangle\langle u_j|$$

$H(\lambda_1, \dots, \lambda_r)$ gives the uncertainty in the the reduced state.

$\mathrm{Tr}_{\mathcal{H}}\, \rho$ gives a state in $\mathcal{K}$ with same eigenvalues $\lambda_1, \dots, \lambda_r$