- More asymptotic notation
- Asymptotics of logs, exponentials, $n!$
- Integral approximations

$f(n) = o(g(n))$ ("little oh")
means $\forall c > 0, \exists N, \forall n \geq N$

$f(n) \leq c\, g(n)$

(equiv: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ )

"f grows strictly slower than g (asymptotically)"

$n = o(n^2)$

$n \leq c \cdot n^2$ for all $n \geq \frac{1}{c}$

Pf: $n \geq \frac{1}{c} \Rightarrow cn \geq 1$

$\Rightarrow cn^2 \geq n$

For all $a < b$, $(a, b \in \mathbb{R})$

$n^a = o(n^b)$

Fact: $\underbrace{f(n) = o(g(n))}_{\text{strict version}} \Rightarrow \underbrace{f(n) = O(g(n))}_{\text{nonstrict asymptotic inequality}}$

$f(n) = O(f(n))$ but
$f(n) \neq o(f(n))$

Def: $f(n) = \omega(g(n))$ means
$\forall c > 0, \exists N, \forall n \geq N,$

$f(n) \geq c\, g(n)$

(equiv: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$ )

(equiv: $g(n) = o(f(n))$ )

Fix constant $b > 1$. $(b \in \mathbb{R})$

$f(n) = b^n$ exponential function to base b

strictly monotone increasing over $\mathbb{R}$

Fact: For every constant k,

$n^k = o(b^n)$

[Proof: $b = e$, $k \in \mathbb{N}$

$\lim_{n\to\infty} \frac{n^k}{e^n} = \lim_{n\to\infty} \frac{kn^{k-1}}{e^n}$

$= \cdots = \lim_{n\to\infty} \frac{k(k-1)\cdots(1)}{e^n} = \lim_{n\to\infty} \frac{k!}{e^n}$

$= k! \lim_{n\to\infty} \frac{1}{e^n} = 0$ ]

$k \notin \mathbb{N}$: $n^k = O(n^{\lceil k \rceil}) = o(e^n)$

$n^k = o(b^n)$

$n^k \leq c b^n$

$\log_b$ of both sides ($\log_b$ is mono increasing because $b > 1$)

$k \log_b n \leq \log_b c + n \log_b b$

$= \underline{\log_b c} + n$

$\therefore \log_b n = o(n)$

Fact: For any real $k > 0$,

$\log_b n = o(n^k)$

$\lg n = o(\sqrt{n})$

Equiv: For any real $\ell > 0$

$(\lg n)^\ell = \lg^\ell n = o(n)$

(take both sides to the power of $\ell := \frac{1}{k}$)

polylog growth

Def: $f(n) = \mathrm{Poly}(g(n))$
means that $\exists k \geq 0$

$f(n) = O(g(n)^k)$

$[\, n^{10} = \underline{\mathrm{Poly}}(n) \,]$

"f(n) is polynomially bounded in g(n)"

$e^n \neq \mathrm{Poly}(n)$

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If $f(n) = O(g(n))$
$\qquad o(g(n))$
$\qquad \Omega(g(n))$
$\qquad \omega(g(n))$

and $h(n)$ is asymptotically positive, then

$f(n)h(n) = O(g(n)h(n))$
$\qquad o(g(n)h(n))$
$\qquad \Omega(g(n)h(n))$
$\qquad \omega(g(n)h(n))$

respectively.

$$\lg n = o(n^{\varepsilon}) \quad \forall \varepsilon > 0$$

$$n^2 \underline{\lg n} = o(n^{2+\varepsilon}) \quad \forall \varepsilon > 0$$

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$$e^x \geq 1 + x \quad (\forall x \in \mathbb{R})$$

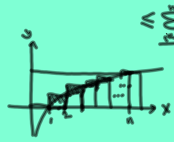
$$e^x \leq 1 + x + x^2 \quad (\forall x,\ |x| \leq 1)$$

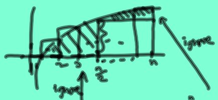
$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1)n$$

$$= \prod_{i=1}^{n} i \qquad \left[0! = \prod_{i=1}^{0} i = 1\right]$$

$$\underline{\lg(n!)} = \lg\left(\prod_{i=1}^{n} i\right) = \sum_{i=1}^{n} \lg i$$

$$\leq \underline{\sum_{i=1}^{n} \lg n} = \underline{n \lg n}$$




$$\sum_{i=1}^{n} \lg i \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \lg i \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \lg \left\lceil \frac{n}{2} \right\rceil$$

$$\geq \frac{n}{2} \cdot \lg\left(\frac{n}{2}\right) = \frac{n}{2}(\lg n - 1)$$

$$\therefore \lg(n!) = \Theta(n \lg n)$$

$$c_1 n \lg n \leq \lg(n!) \leq c_2 n \lg n$$

for some $0 < c_1 \leq c_2$
& all sufficiently large $n$
Take 2 to the power of both sides:

$$2^{c_1 n \lg n} \leq n! \leq 2^{c_2 n \lg n}$$

$$\shortparallel$$

$$(2^{c_1})^{n \lg n} \leq n! \leq (2^{c_2})^{n \lg n}$$

$$1 < b < c \Rightarrow b^n = o(c^n)$$

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Stirling's approximation:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\lim_{n \to \infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1$$

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\alpha_n}$$

where $\dfrac{1}{12n+1} < \alpha_n < \dfrac{1}{12n}$

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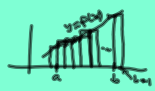
## Integral Approximation

Assume $a, b \in \mathbb{Z}$, $a \leq b$.

1. If $f(x)$ is monotone ascending on $[a-1, b+1]$, then

$$\int_{a-1}^{b} f(x)\,dx \leq \sum_{i=a}^{b} f(i) \leq \int_{a}^{b+1} f(x)\,dx$$




2. If $f(x)$ is monotone descending on $[a-1, b+1]$, then

$$\int_{a}^{b+1} f(x)\,dx \leq \sum_{i=a}^{b} f(i) \leq \int_{a-1}^{b} f(x)\,dx$$

[f is assumed to be Riemann integrable. always true for us]

$$\int_{1}^{n+1} \frac{dx}{x} \leq \sum_{i=1}^{n} \frac{1}{i} \leq \int_{0}^{n} \frac{dx}{x} = \infty$$

$$\boxed{\ln(n+1)} \qquad 1 + \sum_{i=2}^{n} \frac{1}{i} \leq 1 + \int_{1}^{n} \frac{dx}{x} = \boxed{1+\ln}$$

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$$1+\int_2^{n+1}\frac{dx}{x} \le 1 + \sum_{i=2}^{n}\frac{1}{i}$$

$$1+ \ln x\Big|_2^{n+1} = 1 + \ln(n+1) - \ln 2$$

$$= \underbrace{\underbrace{\ln(n+1)} + \underbrace{(1-\ln 2)}_{>0}}$$

How tight? (monotone ascending case)

U.B − L.B = width of the interval containing $\sum_{i=a}^{b} f(i)$

$$\| $$

$$\int_a^{b+1} f(x)\,dx - \int_{a-1}^{b} f(x)\,dx$$

$$= \int_b^{b+1} f(x)\,dx - \int_{a-1}^{a} f(x)\,dx$$

$$\le \int_b^{b+1} \underline{f(b+1)}\,dx - \int_{a-1}^{a} \underline{f(a-1)}\,dx$$

$$= f(b+1) - f(a-1)$$

Iterated lg function

For $n \ge 1$ define

$\lg^* n$ = least $i$ such that $\lg^{(i)} n \le 0$

$$\underbrace{\lg \cdots \lg\, \lg\, \lg}_{i \text{ times}}\, n$$

$$\lg^* 1 = 1$$

$$\lg^* 2 = 2$$

$$\lg 2 = 1$$

$$\underline{\lg \lg} 2 = 0$$

$$\lg^* 3 = 3$$

$$\lg^* 4 = 3$$

$$\lg^* \underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{i} = i+1$$

$$\lg\left(\underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{i}\right) = \underbrace{2^{\cdot^{\cdot^{2}}}}_{i-1}\, \underbrace{\lg 2}_{1} = \underbrace{2^{\cdot^{\cdot^{2}}}}_{i-1}$$

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