Quiz 1 on Thursday 9/6 (not Tuesday 9/4).

Sums $k > -1$

Last time: For $k \geq 0$ (constant)

$$\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$$

$k=-1$? $\sum_{i=1}^{n} \frac{1}{i} = ?$ Harmonic Sum

$$\sum_{i=1}^{n} \frac{1}{i} = \Theta(\lg n)$$

Logarithms: Fix $b>0$ with $b \neq 1$ (base). For any x,

$\log_b x$ is the unique $y \in \mathbb{R}$ such that $b^y = x$

$\left[\begin{matrix} x > 0. \\ x \in \mathbb{R} \end{matrix}\right]$

$\ln = \log_e$

$\lg = \log_2$

$\cancel{\log_b b}^5 = 5$

$\cancel{b}^{\log_b 9} = 9$

Product, quotient, power rules

$x, y > 0$, $r \in \mathbb{R}$

$\log_b(xy) = \log_b x + \log_b y$ (prod rule)

$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$ (quotient rule)

$\log_b(x^r) = r \log_b x$ (power rule)

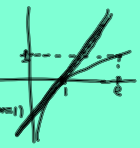
$\log_b 1 = 0$ $\log_b \frac{1}{x} = -\log_b x$

$\log_b b = 1$

$\log_b(x+y) = ?$ no way to expand

Approximations:

$y = \ln x$



$\frac{d \ln x}{dx} = \frac{1}{x} = 1$ $(x=1)$

$= x^{-1}$

$\frac{d^2 \ln x}{dx} = \frac{d(x^{-1})}{dx} = -x^{-2} = -\frac{1}{x^2}$

For all $x > 0$

$\ln x \leq x - 1$

with equality iff $x = 1$.

Prove: $\sum_{i=1}^{n} \frac{1}{i} = \Theta(\underbrace{\lg n}_{\text{base doesn't matter}})$

[change of base rule: $\log_b x = \frac{\log_a x}{\log_a b}$]

$a, b > 1 \Rightarrow \log_a b > 0$

Split the sum more carefully.

Upper bound:

$$\sum_{i=1}^{n} \frac{1}{i} = \underbrace{\frac{1}{1}}_{B_0} + \underbrace{\frac{1}{2} + \frac{1}{3}}_{B_1} + \overbrace{\frac{1}{4} + \frac{1}{5} + \cdots}^{B_2} + \overbrace{\frac{1}{8} + \cdots}^{B_3} + \cdots + \frac{1}{n}$$

$= \underline{B_0 + B_1 + \cdots} + \underline{B_k}$ ← partially filled

$B_i \leq \frac{1}{2^i} + \frac{1}{2^{i+1}} + \cdots + \frac{1}{2^{i+1}-1}$

$\leq \frac{1}{2^i} + \frac{1}{2^i} + \cdots + \frac{1}{2^i}$

$= 2^i\left(\frac{1}{2^i}\right) = 1$

So $B_0 + B_1 + \cdots + B_k \leq k$.

what is k? $\frac{1}{n}$ appears in B_k.

$B_k = \frac{1}{2^k} + \cdots \cancel{+ \frac{1}{2^{k+1}-1}}$ ↑ $\frac{1}{n}$

$2^k \leq n$ lg of both sides

$k \leq \lg n$

$\therefore \sum_{i=1}^{n} \frac{1}{i} \leq \sum_{i=0}^{k} 1 \leq \lg n + 1$

$= O(\lg n)$

Lower bound:

same blocks

$\sum_{i=1}^{n} \frac{1}{i} \geq B_0 + B_1 + \cdots + B_{k-1}$

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each $B_i = \frac{1}{2^i} + \cdots + \frac{1}{2^{i+1}-1}$

$\geq 2^i\left(\frac{1}{2^{i+1}}\right) = \frac{1}{2}$

$\sum_{i=1}^{n} \frac{1}{i} \geq \sum_{i=0}^{k-1} \frac{1}{2} = \frac{k}{2}$

$B_k : \frac{1}{2^k} + \cdots + \frac{1}{2^{k+1}-1}$

$\uparrow$ $1/n$

so $n < 2^{k+1}$

$\therefore \lg n < \cancel{\lg 2^{k+1}} = k+1$

$\therefore k > \lg n - 1$

$\therefore \sum_{i=1}^{n} \frac{1}{i} > \frac{\lg n - 1}{2} = \Omega(\lg n)$

$\lg n$ is eventually positive, unbounded, strictly monotone ascending

$\boxed{x < y \Rightarrow \lg x < \lg y}$

$n < 2^{k+1}$

$\Rightarrow \lg n < \lg 2^{k+1}$

Def: f is monotone ascending iff $x \leq y \Rightarrow f(x) \leq f(y)$

strictly monotone ascending: $x < y \Rightarrow f(x) < f(y)$

f monotone descending iff $x \leq y \Rightarrow f(x) \geq f(y)$

f strictly mont. desc. iff $x < y \Rightarrow f(x) > f(y)$.

Geometric Series

Basic series: $r \in \mathbb{R}$ fixed

$\sum_{i=0}^{n-1} r^i$

$= 1 + r + r^2 + \cdots + r^{n-1}$

finite basic geo. series with ratio r

$\frac{r^{i+1}}{r^i} = r$

$= \frac{r^n - 1}{r - 1}$ if $r \neq 1$, or

if $r = 1$, then $\sum_{i=0}^{n-1} r^i = n$.

$\boxed{r \neq 1:}$

$S := \sum_{i=0}^{n-1} r^i = 1 + r + r^2 + \cdots + r^{n-1}$

$rS = r\sum_{i=0}^{n-1} r^i = \sum_{i=0}^{n-1} r^{i+1} = \sum_{j=1}^{n} r^j$

$j = i+1$

$= r + r^2 + \cdots + r^{n-1} + r^n$

$-$

$rS - S = (r + r^2 + \cdots + r^{n-1} + r^n) - (1 + r + r^2 + \cdots + r^{n-1})$

$(r-1)S = r^n - 1$

$S = \boxed{\frac{r^n - 1}{r - 1}} = \boxed{\frac{1 - r^n}{1 - r}}$

$\sum_{i=0}^{\infty} r^i = \lim_{n\to\infty} \sum_{i=0}^{n-1} r^i$

$= \lim_{n\to\infty}\left(\frac{1 - r^n}{1 - r}\right) = \frac{1}{1 - r}$ ($r^n \to 0$)

provided $|r| < 1$ $(-1 < r < 1)$

(otherwise sum does not converge)

$k < -1$ $\sum_{i=1}^{n} i^k$

$\sum_{i=1}^{n} i^k \leq \sum_{i=1}^{\infty} i^k$ split into blocks as with the harmonic sum

$= B_0 + B_1 + B_2 + \cdots$

where $B_i = (2^i)^k + (2^i + 1)^k + \cdots + (2^{i+1} - 1)^k$

$\leq \underbrace{(2^i)^k}_{\text{biggest term}} \cdot \underbrace{2^i}_{\text{\# of terms}} = (2^i)^{k+1}$

$= 2^{i(k+1)}$

$= (\underbrace{2^{k+1}}_{r})^i$ $r := 2^{k+1}$

$k + 1 < 0$

$0 \therefore < 2^{k+1} < 2^0 = 1$

$0 < r < 1$

$\sum_{i=1}^{\infty} i^k = \sum_{j=0}^{\infty} B_j \leq \sum_{j=0}^{\infty} r^j = \frac{1}{1-r}$

$r = 2^{k+1}$ > 0

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Summary:

$$\sum_{i=1}^{n} i^k \leq \frac{1}{1-r} \quad \text{where } r = 2^{k+1}$$

independent of n.

So $\sum_{i=1}^{n} i^k = \Theta(1)$ as $n \to \infty$

Series of the form

$$\sum_{i=0}^{\infty} i^k r^i$$ where $k \geq 0$ integer, and $|r| < 1$

$k=0$: $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r} = f(r)$

$k=1$: $\sum_{i=0}^{\infty} i r^i = \sum_{i=1}^{\infty} i r^i$

$$\frac{d}{dr} f(r) = \frac{d}{dr} \sum_{i=1}^{\infty} r^i = \sum_{i=1}^{\infty} \frac{d}{dr}(r^i)$$

$$= \sum_{i=1}^{\infty} i r^{i-1}$$

$$\frac{d}{dr} f(r) = \frac{d}{dr}\left(\frac{1}{1-r}\right) = \frac{d}{dr}(1-r)^{-1}$$

$$= (1-r)^{-2} = \underline{\sum_{i=1}^{\infty} i r^{i-1}}$$

$$\boxed{r(1-r)^{-2}} = \sum_{i=0}^{\infty} i r^i$$

$$\boxed{= \frac{r}{(1-r)^2}}$$

$k=2$: $g(r) := \frac{r}{(1-r)^2} = \sum_{i=1}^{\infty} i r^i$

differentiate g in both ways

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