Recall:

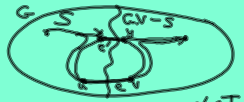
Lemma: G a weighted (undir) graph with n vertices & m edges. $A \subseteq G.E$ and A respects some cut $(S, G.V-S)$ $(S \subseteq G.V)$. Let $e \in G.E$ cross cut $(S, G.V-S)$ with min weight (among all edges crossing this cut). If A is a subset of some MST of G, then so is $A \cup \{e\}$.

Proof: A, e, $(S, G.V-S)$ as above. Let $T \subseteq G.E$ be a MST for G s.t. $A \subseteq T$.

Case 1: $e \in T$. Done.

Case 2: $e \notin T$. Consider $T \cup \{e\}$. T has $n-1$ edges, so $T \cup \{e\}$ has n edges,

$\therefore$ it contains a cycle including edge e.



$e=(u,v)$. $u \in S$, $v \notin S$. $e' \in T$

Let $T' = (T - \{e'\}) \cup \{e\}$

Claim: T' is a spanning tree.

T' is connected. Why?

T' has $n-1$ ~~vertices~~ edges So

T' is a spanning tree.

$$\begin{aligned} cost(T') &= cost(T) - wt(e') + wt(e) \\ &= cost(T) - \underbrace{(wt(e') - wt(e))}_{\geq 0} \\ &\leq cost(T) \end{aligned}$$

T is an MST, so $cost(T') = cost(T)$

$\therefore T'$ is an MST & ~~$e \in T'$~~

$A \cup \{e\} \subseteq T'$ ▨

Single source Shortest Paths

Bellman-Ford algo.

Allows negative edge weights (no negative cycles)

Dijkstra's Algo. Faster than Bellman-Ford but no negative edge weights allowed.

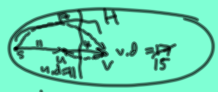
G is a weighted digraph G, all edge wts ≥ 0.

Input ~~vertex~~ $s \in G.V$ (source vertex)

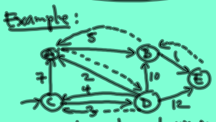
Dijkstra: Find min cost paths from s to all vertices, along with cost "cost" = distance (shortest paths/distances).

Dijkstra:

```
for all v∈G.V except s,
    set v.d:=∞
        v.π:=nil
s.d:=0
s.π:=nil
Put all vertices into a minheap H
    d-attribute is key.
while H is not empty, do
s at first  //remove a vertex u from H
        (u has min d-attr — ExtractMin)
    for each vx v adjacent from u (∈H)
        if v.d > u.d + wt(u,v) then
            v.d := u.d + wt(u,v)
                //decreasekey
            v.π := u
    end for each
end while
```



Example:



d- & π-values when each vx is removed from H. Source = C

A	B	C	D	E
∞, nil	∞, nil	0, nil	∞, nil	∞, nil
7, C	∞, nil	"	3, C	∞, nil
5, D	13, D	"	"	15, D
"	10, A	"	"	15, D
"	"	"	"	11, B

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Running time of Dijkstra
= Running time of Prim
(asymptotically)

$= \Theta(\underbrace{|E| + |V| \lg |V|}_{\text{if } H \text{ is a Fib heap}})$

[otherwise $\Theta(|E| \lg |V|)$]

Correctness of Dijkstra's algo.

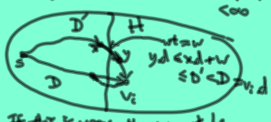
Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices in order of heap removal ($v_1 = s, \ldots$)

Prove by induction on i that final $v_i.d$ is shortest distance $s \to v_i$ and $v_i.\pi$ is predecessor along some shortest path $s \to v_i$.

(Remember: all edge wts ≥ 0)

$i = 1$: $v_1 = s$. $s.d = 0$, $s.\pi = nil$ finally. Correct.

Now assume $i > 1$ and all values $v_j.d$ & $v_j.\pi$ for $j < i$ are correct upon removal from H (final values). Consider v_i, when it is removed from H. Suppose that $v_i.d$ is wrong at this time. Let $D = v_i.d$. $< \infty$



If this is wrong, there must be a shorter path $s \to v_i$ of length $D' < D$.

Claim: $y.d \leq D'$

So $y.d < v_i.d$, so v_i's key not minimum in H, so v_i not removed from H at this point. $\rightarrow\leftarrow$

Floyd-Warshall algo:
all pairs shortest path
Time is $\Theta(|V|^3)$?

NP-completeness

Polynomial Reduction

Def: A decision problem is one where the output is binary (single bit, e.g. "yes" or "no") for any given input.

Usually specify a decision problem as follows

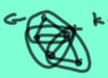
1. Describe the input (instance)
2. Describe a yes/no question about the instance.

Ex: CLIQUE

Input: A graph G and an integer k.

Question: Does G have a clique of size $\geq k$?

(a clique is a set of vertices such that G includes edges between all pairs of vertices in the clique)

G  $k = 3$ yes $k = 4$ no

Def: Let Π_1 & Π_2 be decision problems. A polynomial reduction from Π_1 to Π_2 is a function f mapping any instance of Π_1 to an instance of Π_2 such that

1. f is computable in time $O(n^k)$ for some constant k ("polynomial time") [n is size of Π_1 instance]
2. f maps yes-instances to yes-instances and no-instances to no-instances.

I.e., X and $f(X)$ have the same answer.

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