Quiz 11:

i th op costs 1 – if i not perfect square

$\sqrt{i}$ if i is a perfect square

i	1	2	3	4	5	6	7	8	9	10
c_i	1	1	1	2	1	1	1	1	3	1 ⋯

Aggregate analysis:

Total time

$$T = \sum_{i=1}^{n} c_i$$

$$= \sum_{i=1}^{n} 1 + \sum_{\substack{k=1 \\ k^2 \le n \\ k = \sqrt{k^2}}}^{\lfloor \sqrt{n} \rfloor} (k - 1)$$

$$\le n + \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} k$$

$$\le n + \frac{\sqrt{n}(\sqrt{n}+1)}{2}$$

$$= n + \frac{n}{2} + \frac{\sqrt{n}}{2} = O(n)$$

divide by n — get $O(1)$ amortized cost per op.

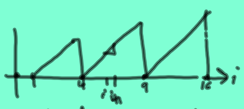
potential method:

$\Phi_0 := 0$

$i > 0$: define Φ_i as follows:

Let k be greatest such that $k^2 \le i$ $(k = \lfloor \sqrt{i} \rfloor)$

Set $\Phi_i := (i - k^2)$



Amortized cost for ith op:

$$\hat{c}_i = c_i + \Phi_i - \Phi_{i-1}$$

Case 1: i is not a perfect square. Then $\Phi_i - \Phi_{i-1} = 1$

$$\hat{c}_i = 1 + 1 = 2 = O(1)$$

Case 2: i is a perfect square.

Then $\Phi_i = 0$.

Say $i = k^2$ for some k

$$\Phi_{i-1} = i - 1 - (k-1)^2$$

$$= k^2 - 1 - (k-1)^2$$

$$= \cancel{k^2} - 1 - \cancel{k^2} + 2k - 1$$

$$= 2k - 2 = 2(k-1)$$

so $\hat{c}_i = \underset{\sqrt{i}}{\underset{\shortparallel}{k}} + \underset{\Phi_i}{\underset{\shortparallel}{0}} - \underset{\Phi_{i-1}}{\underset{\shortparallel}{2(k-1)}}$

$$= k - 2k + 2$$

$$= 2 - k \le 2 = O(1)$$

Resizable Arrays (cont.)

Array A with capacity n

$A[1..n]$

and size s — s items currently in A. $(s \le n)$

Define load factor

$$\alpha = \frac{s}{n} \quad 0 \le \alpha \le 1$$

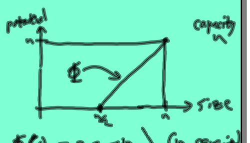
size doubling when $s > n$

$n \longrightarrow 2n$ capacity

$n \longrightarrow n+1$ size

$\alpha: 1 \longrightarrow \frac{n+1}{2n} \doteq \frac{1}{2}$

resize $n \to 2n$ costs n

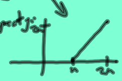


$\Phi_n(s) = 2s - n$ (n capacity)

for $\frac{n}{2} \le s \le n$

after $n \to 2n$ capacity:

$\Phi_{2n}(s) = 2s - 2n$



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resize $n \longrightarrow 2n$

cost $= n$ (recopying)

$\Delta\Phi = \Phi_{2n}(n) - \Phi_n(n)$

$= 0 - n = -n$

amortized cost of a resize is

$c_i + \Delta\Phi = n - n = 0$

amortized cost of an insertion (without resizing)

$= \underset{\substack{\uparrow \\ \text{actual cost}}}{1} + \Phi_n(s) - \Phi_n(s-1)$

$= 1 + 2 = 3$

$\therefore O(1)$ amortized cost per op.

Insert, Delete are allowed
1 1 unit cost

downsize array to save space

upsizing strategy can be the same
(double capacity on overflow)

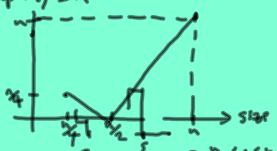
downsize strategy: halve the capacity when the load factor would drop below $\frac{1}{4}$ on the next deletion.

actual cost of downsizing $n \Rightarrow \frac{n}{2}$
$= \frac{n}{4}$

Load factor after a downsize is $\frac{1}{2}$. $\Rightarrow \Phi_{new} = 0$

Now $\frac{1}{4} \leq \alpha \leq 1$

capacity $= n$



$$\Phi_n(s) = \begin{cases} 2s - n & \text{if } \frac{n}{2} \leq s \leq n \\ \frac{n}{2} - s & \text{if } \frac{n}{4} \leq s \leq \frac{n}{2} \end{cases}$$

amortized cost of a downsize deletion:

$= \underset{\text{actual cost}}{\frac{n}{4}} + \underbrace{\Phi_{\frac{n}{2}}\left(\frac{n}{4}\right)}_{0 \text{ because } \alpha = \frac{1}{2}} - \underbrace{\Phi_n\left(\frac{n}{4}\right)}_{n/4}$

$= \frac{n}{4} + 0 - \frac{n}{4} = 0$

cost of a deletion without downsizing is $\underset{\text{actual cost}}{1} + \underset{\Delta\Phi \leq \cancel{2}\ 1}{O(1)}$

Fibonacci Heaps

Implementation of a Mergeable heap (abstract data struct)

A mergeable heap is any collection that supports the following ops:

- Insert(H, x)
- FindMin(H)
- ExtractMin(H)
- MakeHeap() — new empty heap
- Union(H_1, H_2) — merges H_1, H_2 into a single heap H (returned). H_1, H_2 are "destroyed" in the process

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