Today:

- Indicator r.v.'s
- Hiring problem
- Randomized quicksort

Quiz 3: Substitution method

If $T(n) = 7T(\lfloor \frac{n}{8} \rfloor) + n$
then $T(n) = O(n)$.

Prove by induction that
$T(n) \le cn$. (constant $c>0$)

$T(n) = 7T(\lfloor \frac{n}{8} \rfloor) + n$

$\le 7c\lfloor \frac{n}{8} \rfloor + n$

$\le 7c(\frac{n}{8}) + n$

$= n(\frac{7}{8}c + 1)$

$\le cn$ provided

$\frac{7}{8}c + 1 \le c.$

$\Updownarrow$

$1 \le c - \frac{7}{8}c$

$1 \le \frac{c}{8}$

$\boxed{8 \le c}$

So $T(n) \le cn$ if $c \ge 8$.

Quiz 5

$\Pr[\text{die values all distinct}]$

$= \frac{\#\text{ distinct sequences}}{\#\text{ all possible sequences}}$

$= \frac{\cancel{6} \cdot 5 \cdot 4}{\cancel{6} \cdot 6 \cdot 6} = \frac{5}{9}$

success probability $p = \frac{5}{9}$

Geometric distribution

$N = \#$ of trials until success

we want $E[N]$

$= \frac{1}{p} = \boxed{\frac{9}{5} = 1.8}$

$\left[\frac{1}{p} - 1 = \frac{4}{5} = 0.8\right]^{\text{possibly}}$

$E[N] = \sum_{k=1}^{\infty} \Pr[N=k] \cdot k$ $\boxed{q = 1-p}$

$= \sum_{k=1}^{\infty} q^{k-1} p k$

$= p \sum_{k=0}^{\infty} (k+1) q^k$

$= p\left(\underline{\sum_{k=0}^{\infty} k q^k} + \underline{\sum_{k=0}^{\infty} q^k}\right)$

$\boxed{q<1}$

$= p\left[\frac{q}{(1-q)^2} + \frac{1}{1-q}\right]$

$= p\left[\frac{1-p}{p^2} + \frac{1}{p}\right]$

$= \frac{1-p}{p} + 1 = \frac{1-p+p}{p}$

$= 1/p.$

Indicator r.v.'s

Fix prob space Ω

Event $A \subseteq \Omega$.

$\underbrace{I\{A\}}_{\text{indicator r.v. of event } A.}(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$

$E[I\{A\}] = \sum_{x \in \Omega} \underbrace{I\{A\}(x)} \cdot \Pr[x]$

$= \sum_{x \in A} \Pr(x) = \Pr[A].$

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Hiring problem

Candidates ranked $1, \ldots, n$

n = best ; 1 = worst

Assume chronological order is a random permutation of $\{1, \ldots, n\}$.

$\Omega = \{\sigma : \sigma$ is permutation of $1, \ldots, n\}$

$|\Omega| = n!$ $\quad Pr[\sigma] = \frac{1}{n!}$

(uniform distribution)

H = # of hirings

$H(\sigma)$ = # of hirings with order σ.

Think of $\sigma = \langle r_1, r_2, \ldots, r_n \rangle$

r_1 comes first

r_2 second

$\vdots$

r_n last

$\langle r_1, \ldots, r_n \rangle$ is a permutation of $\langle 1, 2, \ldots, n \rangle$

$H(\sigma) = \# \; i \in \{1, \ldots, n\}$:

$\underline{r_i} > r_j$ for all $j < i$.

Best case: 1 hiring

(best candidate first, or $r_1 = n$)

Worst case: n: $\quad r_i = i$ for all i

Average case is $E[H]$.

For $i = 1$ to n, let

X_i be the event

"ith candidate is hired"

= "$r_i > r_j \; (\forall j < i)$"

$$I\{X_i\} = \begin{cases} 1 & \text{if } i \text{ is hired} \\ 0 & \text{o.w.} \end{cases}$$

$H = I\{X_1\} + I\{X_2\} + \cdots + I\{X_n\}$

$= \sum_{i=1}^{n} I\{X_i\}$

so

$$E[H] = E\left[\sum_{i=1}^{n} I\{X_i\}\right]$$

$$= \sum_{i=1}^{n} E\left[\underline{I\{X_i\}}\right]$$

$$= \sum_{i=1}^{n} \underbrace{Pr[i\text{th cand. is hired}]}$$

Fix i $\quad (1 \le i \le n)$

$Pr[i\text{'th cand. hired}] = \frac{1}{i}$

why?

$Pr[\quad " \quad]$

$= Pr[r_i$ is $\max\{r_1, \ldots, r_i\}]$

$= \frac{1}{i}$ because maximum could be any of $r_1, \ldots, r_i$ with equal probability.

$$E[H] = \sum_{i=1}^{n} \frac{1}{i} = \ln n + O(1)$$

$$= \Theta(\lg n)$$

Randomized Quicksort

Input: $A[1..n]$ ← numbers, p, q

Output: $A[1..n]$ where subarray $A[p..q]$ is sorted (rest of array unchanged)

Top-level call:

QuickSort($A, 1, n$)

QuickSort(A, p, q)

QuickSort(A, p, q):

if $p < q$ then

$m := \text{Partition}(A, p, q)$

// m is the index

// of the pivot in $A[p..q]$

// $p \le m \le q$

QuickSort(A, p, m−1)

QuickSort(A, m+1, q)

Randomized strategy for finding a pivot: choose pivot uniformly at random from $A[p..q]$ (by index)

$q-p+1$ items; each can be pivot with prob $\frac{1}{q-p+1}$

Running time of randomized quicksort is a random variable

Sample space is over all possible sequences of random choices made by the algo.

T running time r.v.

We want average case

$E[T]$.

Best case: $\Theta(n \lg n)$

pivot of $A[p..q]$ always close to the median of $A[p..q]$

Worst case: $\Theta(n^2)$

pivot is always an extreme value of $A[p..q]$.

Find a recurrence satisfied by $E[T]$. Only look for an upper bound. Assume that $E[T]$ is monotone in n. ascending.