Random experiment :

Sample space : set S of possible outcomes

S (Ω)

Fair coin flip :

$S = \{H, T\}$

A probability distribution Pr on S maps events to real numbers

An <u>event</u> is any subset of S.

<u>discrete</u> probability: S is finite (or countable)

Pr satisfies

1. $\Pr[A] \geq 0$ (every $A \subseteq S$)
2. $\Pr[S] = 1$
3. If $A, B \subseteq S$ & $A \cap B = \emptyset$, then $P[A \cup B] = \Pr[A] + \Pr[B]$

If $A_1, A_2, A_3, \ldots$ are pairwise disjoint events, then

$$\Pr\left[\bigcup_i A_i\right] = \sum_i \Pr[A_i].$$

Experiment produces an <u>outcome</u> (element of S)

$\Pr[A]$ = likelyhood that outcome is in A.

Toss 2 coins

$S = \{HH, HT, TH, TT\}$

$A \subseteq S$: "at least one head"

$A = \{HH, HT, TH\}$

Fair coin :

$\Pr[H] = \Pr[T] = \frac{1}{2}$

$\Pr[\emptyset] = 0$ | By additivity.

$\Pr[S] = 1$ |

$$\Pr[A] = \sum_{x \in A} \underbrace{\Pr[\{x\}]}_{\Pr[x]}$$

Only need to specify $\Pr[x]$ for all $x \in S$.

$\Pr[\emptyset] = 0$ "impossible event"

$\Pr[S] = 1$ <u>"S is the certain event"</u>

S finite, k elements

<u>Uniform</u> distribution on S assigns the same value $1/k$ to every outcome in S.

$$\Pr[A] = \sum_{s \in A} \Pr[s] = \sum_{s \in A} \frac{1}{k} = \frac{|A|}{k} = \frac{|A|}{|S|}$$

2 fair coin flips

uniform distribution on $S = \{HH, HT, TH, TT\}$

$\Pr[\geq 1 \text{ head}] = \frac{3}{4}$

$A \subseteq B \Rightarrow \Pr[A] \leq \Pr[B]$ (any distribution)

$$\Pr[\underbrace{S - A}_{\overline{A}}] = 1 - \Pr[A]$$

For all events A, B

$\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$

Conditional probability & independence

Fix an event $B \subseteq S$ s.t. $\Pr[B] > 0$.

For any event A,

$$\underbrace{P[A \mid B]}_{\substack{\text{prob dist} \\ \text{on } S \\ (B \text{ fixed})}} := \frac{\Pr[A \cap B]}{\Pr[B]}$$

New sample space B

$$\Pr[\underline{2 \text{ heads}} \mid \underline{\geq 1 \text{ head}}] = \frac{1/4}{3/4} = 1/3$$

Two events A,B are <u>independent</u>

iff $Pr[A \cap B] = Pr[A]Pr[B]$

A,B are mutually exclusive if

$Pr[A \cap B] = 0$.

If $Pr[B] > 0$,

A,B independent iff

$Pr[A] = Pr[A|B]$

$Pr[A] > 0$: $Pr[B] = Pr[B|A]$

A = "first flip heads" — $1/2$

B = "second flip tails" — $1/2$

$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

A = "the two flips are different"

$= \{HT, TH\}$

B = "second flip tails"

A,B independent.

$Pr[A \cap B] > Pr[A]Pr[B]$

means A & B are positively correlated

if $<$, then negatively correlated.

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2 rolls of a standard 6-sided die

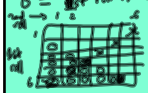
$S = \{(1,1), (1,2), (1,3), \ldots, (1,6),$
$(2,1), (2,2), \ldots, (2,6),$
$\vdots$
$(6,1), (6,2), \ldots, (6,6)\}$

36 outcomes uniformly distributed

$Pr[(x,y)] = \frac{1}{36}$.

A = "rolls sum to 7"

B = "1st roll is > 2nd roll"



$Pr[A] < \frac{1}{6}$

$Pr[B] = \frac{5}{12}$

$Pr[A \cap B] = \frac{1}{12}$

$Pr[A] \cdot Pr[B] = \frac{1}{12}\left(\frac{5}{6}\right) < \frac{1}{12}$

Random variables: (r.v.'s)

r.v. $X : S \to \mathbb{R}$

$X(a)$ is numerical value associated with outcome $a \in S$.

2 dice example:

X = sum of the two rolls

Y = smaller of the two rolls

Z = 1st roll

etc. X r.v.

$x \in \mathbb{R}$, we let

$Pr[X = x] \geq 0$

$:= Pr\{\{a \in S : X(a) = x\}\}$

$\sum_x Pr[X = x] = 1$

E.g. $Pr[X \geq x]$

$= Pr[\{a : X(a) \geq x\}]$

$= \sum_{y \geq x} Pr[X = y]$

X fixed r.v.

$f(x) = Pr[X = x]$

<u>probability density function</u> for X

Expected value of X

$E[X] = \sum_x x f(x)$

$= \sum_x x \cdot Pr[X = x]$

$E[X+Y] = E[X] + E[Y]$

$E[aX] = aE[X]$

($a \in \mathbb{R}$ constant)

$X_1, \ldots, X_n$ r.v.'s over S

$a_1, \ldots, a_n \in \mathbb{R}$ then

$E\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i E[X_i]$

linearity of expectation

2 dice example:

X = min of the two rolls

$E[X] = ?$

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$E[X]$

	1					6
1	1	1	1	1	1	1
	1	2	2	2	2	2
	1	2	3	3	3	3
	1	2	3	4	4	4
	1	2	3	4	5	5
6	1	2	3	4	5	6

x	$\Pr[X=x]$
1	11/36
2	9/36 = 1/4
3	7/36
4	5/36
5	3/36 = 1/12
6	1/36

$$E[X] = 1\cdot\frac{11}{36} + 2\cdot\frac{9}{36} + 3\cdot\frac{7}{36} + 4\cdot\frac{5}{36} + 5\cdot\frac{3}{36} + 6\cdot\frac{1}{36}$$

$$= \frac{1}{36}(11 + 18 + 21 + 20 + 15 + 6)$$

$$= \frac{91}{36} \doteq 2.5\ldots$$

Variance — measures spread of an r.v.

$$V[X] := E\left[(X - \underline{E[X]})^2\right]$$

standard deviation

$$\sigma := \sqrt{V[X]}$$

$$V[X] = E[X^2] - E[X]^2$$
$$= E[X^2] - \mu^2$$

where $\mu = E[X]$

Binomial distribution

n identical binary trials

2 outcomes
success,
failure

(flipping a biased coin n times)

p = success probability

q = failure prob $= 1-p$

N = the number of successes among n trials

$$\Pr[N=k] = \binom{n}{k} p^k q^{n-k}$$

N is binomially distributed

$(0 \le k \le n)$

$$E[N] = pn$$

$$V[N] = pqn$$

Geometric distribution

$$S = \{s, fs, ffs, fffs, \ldots, f^k s, \ldots\}$$

p = success prob $\quad q = 1-p$

$$\Pr[f^k s] = q^k p$$

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