Quiz ~~8~~ 4 solution

Proof sketch of Master Thm

$T(n) = \Theta(n^2)$

Master Method:

$T(n) = 2\,T\left(\frac{7n}{10}\right) + n^2$

$a = 2$ $\quad t = \log_b a$

$b = 10/7$ $\quad = \underbrace{\log_{\frac{10}{7}} 2}$ vs 2

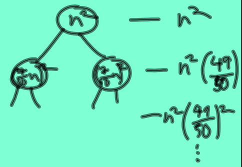
$f(n) = n^2$

Case 3: $\quad \left(\frac{10}{7}\right)^x = 2$

$T(n) = \Theta(f(n))$ $\quad \therefore\ t < 2$

$= \Theta(n^2)$.

Tree method:



$n^2 \le T(n) \le \sum_{i=0}^{depth} n^2\left(\frac{49}{50}\right)^i$

$= n^2 \sum_{i=0}^{depth} \left(\frac{49}{50}\right)^i$

$\le n^2 \underbrace{\sum_{i=0}^{\infty} \left(\frac{49}{50}\right)^i}$

$= n^2\left(\frac{1}{1-\frac{49}{50}}\right)$

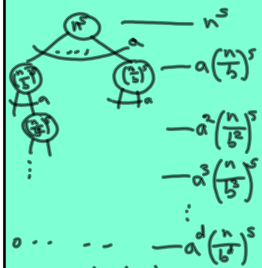
$= n^2 (50)$

$T(n) = \Theta(n^2)$.

Sketch proof of Master Thm using tree method

Special case: $f(n) = n^s$ (constant $s \ge 0$).

$T(n) = a\,T\left(\frac{n}{b}\right) + n^s$



where d = depth of the tree

$d = \log_b n$

$T(n) = \sum_{i=0}^{d} a^i\left(\frac{n}{b^i}\right)^s$

$= n^s \sum_{i=0}^{d} \frac{a^i}{b^{is}}$

$= n^s \sum_{i=0}^{d} \left(\frac{a}{b^s}\right)^i$

$= n^s \sum_{i=0}^{d} r^i$ where $r = \frac{a}{b^s}$

Case 2: $r = 1$:

$b^s = a$ $\quad \log_b$ of both sides

$s = s\log_b b = \log_b a = t$

(case 2 of master Thm)

$T(n) = n^s(d+1)$

$= \Theta(n^s d)$

$= \Theta(n^s \log_b n)$

$= \Theta(n^s \lg n)$

all levels equal.

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$T(n) = n^s \sum_{i=0}^{d} r^i \qquad r = \frac{a}{b^s}$

<u>Case 3</u>: $r < 1$

$\frac{a}{b^s} < 1 \longrightarrow a < b^s$

$t = \log_b a < s$

(case 3 of Master Thm)

$n^s \le T(n) = n^s \sum_{i=0}^{d} r^i$

$\le n^s \underbrace{\sum_{i=0}^{\infty} r^i}_{\frac{1}{1-r} > 0} = O(n^s)$

$\therefore T(n) = \Theta(n^s) = \Theta(f(n))$

tree is top-heavy.

Case 1: $r > 1$.

$T(n) = n^s \sum_{i=0}^{d} r^i$

$= n^s \left(\frac{r^{d+1} - 1}{r - 1} \right)$

$= \Theta(n^s r^{d+1})$

$= \Theta(n^s r^d)$

$\boxed{d = \log_b n}$

$= \Theta(n^s r^{\log_b n})$

$= \Theta(n^s n^{\log_b r})$

$= \Theta(n^{s + \log_b r})$

$[r = \frac{a}{b^s}]$

$= \Theta\left(n^{s + \log_b \left(\frac{a}{b^s}\right)}\right)$

$= \Theta\left(n^{s + \log_b a - \underbrace{\log_b b^s}_{s}}\right)$

$= \Theta(n^{\log_b a})$

$= \Theta(n^t)$.

tree is bottom-heavy.

— n^s

— $r n^s$

— $r^2 n^s$

—

Counting things

(Appendix C)

A B, disjoint

$|A|$ ways to select from A

+

$|B|$ " " " " B

$|A| + |B|$ possible ways to select an element from either A or B

(<u>sum rule</u>)

(<u>product rule</u>) A, B arbitrary

$|A| \cdot |B|$ ways to select a pair <u>(a, b)</u> where $a \in A$ and $b \in B$

$|A \times B|$

k labeled elements from A

$(1, 2, 3, \ldots, k)$

with replacement

$|A|^k$ ways (iterated product rule)

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A, B sets

$|A| = n$

$|B| = m$

How many functions from A into B?

m^n many ways

$A = \{a_1, a_2, \ldots, a_n\}$

functions = # n-tuples from B,

product rule says $= |B|^n = m^n$.

"without replacement"

A

select a sequence of k labeled elements from A, all distinct.

$$\underbrace{n(n-1)(n-2)\cdots(n-k+1)}_{k \text{ factors}}$$

$$= \frac{n!}{(n-k)!}$$

$$k = n: \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

$n!$ = # permutations of A

= # possible ways of (re)ordering the elements of A

A, B $\quad |A| = n,\ |B| = m$

How many 1-1 functions from A into B?

$$= \frac{m!}{(m-n)!} \quad (m \geq n)$$

(if $m < n$, then $\emptyset$)

A permutation of A is a 1-1 function of A onto A

($n!$ of these)

$|A| = n$

k ~~unlabeled~~ distinct elements of A, without regard to order.

counting the # subsets of A with k elements.

$$\frac{n!}{(n-k)!\,k!} = \underbrace{\binom{n}{k}}_{\text{Binomial coefficient}} \quad \text{"n choose k"}$$

If order does matter, then we saw $\frac{n!}{(n-k)!}$ many ways;

this counts the same collection of k elements exactly $k!$ many times (one for each ordering of the k elements). So,

$$\frac{n!}{(n-k)!} = k!\,(\text{\# of } k\text{-subsets of } A)$$

$$\left(\begin{array}{l}\text{\# of}\\ k\text{-subsets}\\ \text{of } n \text{ elements}\end{array}\right) = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

$$0 \leq k \leq n$$

<u>Binomial Thm:</u>

For any numbers x, y and integer $n \geq 0$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

So in particular

$x = y = 1$:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$x = 1,\ y = -1$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots = 0$$

(if $n > 0$).

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