More on substitution method

$T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + n$

Guess that $T(n) = O(n \lg n)$.

Prove by induction that

$[\underline{T(n)} \leq \underline{c n \lg n}]$

(some constant $c > 0$)

$T(n) = 2\underbrace{T(\lfloor \frac{n}{2} \rfloor)}_{<n} + n$

$\leq 2c \lfloor \frac{n}{2} \rfloor \lg \lfloor \frac{n}{2} \rfloor + n$

$\leq \cancel{2} c \frac{n}{\cancel{2}} \lg \frac{n}{2} + n$

$= c n \lg \frac{n}{2} + n$

$= c n (\lg n - 1) + n$

$= c n \lg n + n - c n$

$\leq c n \lg n$

provided $c \geq 1$ ✓

Lower bound:

$T(n) = \Omega(n \lg n)$

$(\therefore T(n) = \Theta(n \lg n))$

$[T(n) \geq c n \lg n]$

prove by induction

$T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + n$

$\geq 2c \lfloor \frac{n}{2} \rfloor \lg \lfloor \frac{n}{2} \rfloor + n$

$\lfloor x \rfloor > x - 1$

$\geq 2c \underbrace{(\frac{n}{2} - 1)}_{\geq n/4} \lg (\frac{n}{2} - 1) + n$

$\cancel{\geq c (\frac{n}{2}) \lg (\frac{n}{4}) + n}$

$\cancel{= \frac{cn(\lg n - 2)}{2} + n}$

$\times$

$\cancel{\geq c n \lg n}$

$2c(\frac{n}{2} - 1) \lg \underbrace{(\frac{n}{2} - 1)}_{\geq n/4} + n$

$\geq 2c(\frac{n}{2} - 1) \lg (\frac{n}{4}) + n$

$= 2c(\frac{n}{2} - 1)(\lg n - 2) + n$

$= \underline{c n \lg n} - 2c \lg n - 2cn + 4c + n$

$\geq c n \lg n$

provided

$-2c \lg n - 2cn + 4c + n \geq 0$

$\| $

$n(1 - 2c) - 2c \lg n + 4c$

$n(1 - 2c) - 2c \lg n$

$\geq n(1 - 2c) - 2c \frac{n}{10}$

(provided n is large enough)

$= n(\underline{1 - 2c - \frac{c}{5}})$

≥ 0 provided

$1 - 2c - \frac{c}{5} \geq 0$

i.e., $1 \geq c(2 + \frac{1}{5})$

$\frac{5}{11} \geq c$

Sep 9-12:31 PM

Thm: If $T(n) = 2T(\frac{n}{2}) + n$

floor or ceiling

then $T(n) = \Theta(n \lg n)$

$\lceil x \rceil < x + 1$

A subtlety

$T(n) = T(\lfloor \frac{n}{2} \rfloor) + T(\lceil \frac{n}{2} \rceil) + 1$

guess that $T(n) = \Theta(n)$.

Try proving (upper bound)

$T(n) \leq cn$ (constant c)

$$
\begin{aligned}
T(n) &= T(\lfloor \tfrac{n}{2} \rfloor) + T(\lceil \tfrac{n}{2} \rceil) + 1 \\
&\leq c\lfloor \tfrac{n}{2} \rfloor + c\lceil \tfrac{n}{2} \rceil + 1 \\
&= c(\lfloor \tfrac{n}{2} \rfloor + \lceil \tfrac{n}{2} \rceil) + 1 \\
&= cn + 1 \\
&\not\leq cn \quad \text{whoops.}
\end{aligned}
$$

Instead, try proving by induction that

$T(n) \leq cn - d$ $(= O(n))$

(constants $c > 0$, $d \geq 0$)

$$
\begin{aligned}
T(n) &= T(\lfloor \tfrac{n}{2} \rfloor) + T(\lceil \tfrac{n}{2} \rceil) + 1 \\
&\leq c\lfloor \tfrac{n}{2} \rfloor - d + c\lceil \tfrac{n}{2} \rceil - d + 1 \\
&= cn - 2d + 1 \\
&\leq cn - d
\end{aligned}
$$

provided $-d + 1 \leq 0$

i.e. $d \geq 1$

$T(n) = 2T(\lfloor \frac{n}{2} \rfloor + 17) + n$

prove that $T(n) = \Theta(n \lg n)$

Hint: upper bound:

u.b. $(\lfloor \frac{n}{2} \rfloor + 17)$ by

$\frac{n}{2} + 18$ (to remove the floor)

outside the logarithm

inside the log, upper bound

$\lfloor \frac{n}{2} \rfloor + 17$ by $\frac{2}{3} n$

(true if n is large enough)

Alternate approach:

Prove that

$T(n) \leq cn \lg(n - 34)$.

Change of variables

$T(n) = 2T(\sqrt{n}) + \lg n$

Define

$S(m) := T(2^m)$

$n \longleftrightarrow 2^m$

$$
\begin{aligned}
S(m) &= T(2^m) \\
&= 2T(\sqrt{2^m}) + \lg(2^m) \\
&= 2T((2^m)^{1/2}) + \lg(2^m) \\
&= 2T(2^{m/2}) + m \\
&= 2S(m/2) + m
\end{aligned}
$$

Sep 9-12:56 PM

$S(m) = 2S(\frac{m}{2}) + m$

so $S(m) = \Theta(m \lg m)$

$n = 2^m$

$\lg n = m$

so

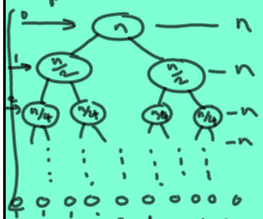
$T(n) = S(\lg n)$

$= \Theta(\lg n \lg \lg n)$

Tree method:

$T(n) = 2T(\frac{n}{2}) + n$

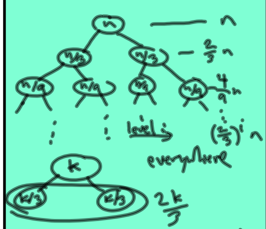
Input of size n



$T(n)$ = sum of nodes in the tree.

There are $\lg n + 1 = \Theta(\lg n)$ levels, each level is n, so total time is $\Theta(n \lg n)$

$T(n) = 2T(\frac{n}{3}) + n$



Total time (upper bound)

$T(n) = \sum_{i=0}^{depth} n(\frac{2}{3})^i$

$= n \sum_{i=0}^{depth} (\frac{2}{3})^i$

$\leq n \sum_{i=0}^{\infty} (\frac{2}{3})^i$

$= n\left(\frac{1}{1-\frac{2}{3}}\right) = 3n$

$\therefore T(n) = O(n)$

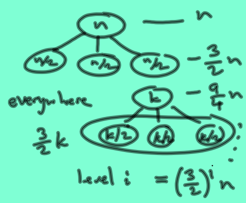
$T(n) = 2T(\frac{n}{3}) + n$

$T(n) = \Omega(n)$

obviously

$T(n) = \Theta(n)$

$T(n) = 3T(\frac{n}{2}) + n$



$T(n) = \sum_{i=0}^{depth=d} (\frac{3}{2})^i n$

$= n \sum_{i=0}^{depth=d} (\frac{3}{2})^i$

$= n\left(\frac{(\frac{3}{2})^{d+1} - 1}{\frac{3}{2} - 1}\right)$

$= \Theta\left(n (\frac{3}{2})^{d+1}\right)$

$= \Theta\left(n (\frac{3}{2})^{d}\right)$

$$T(n) = \Theta\left(n\left(\tfrac{3}{2}\right)^d\right)$$

$d = \lg n$:

$$T(n) = \Theta\left(n\left(\tfrac{3}{2}\right)^{\lg n}\right)$$

[Recall: $a^{\log_b c} = c^{\log_b a}$]

Proof: $\log_b a = p$

$a = b^p$

l.h.s $= b^{p \log_b c} = (\underbrace{b^{\log_b c}}_{c})^p$

$= c^p = c^{\log_b a}$.

$$T(n) = \Theta\left(n\left(\tfrac{3}{2}\right)^{\lg n}\right)$$

$$= \Theta\left(n \cdot n^{\lg\left(\frac{3}{2}\right)}\right)$$

$\lg \frac{3}{2} = \lg 3 - 1$

So $T(n) = \Theta(n^{\lg 3})$

$1 = \lg 2 < \lg 3 < \lg 4 = 2$

$$T(n) = T\left(\tfrac{n}{3}\right) + T\left(\tfrac{2n}{3}\right) + n$$

$$T(n) = T\left(\tfrac{2n}{3}\right) + T\left(\tfrac{3n}{4}\right) + n^2$$

Sep 9-1:35 PM