- Integral approximations
- little o, ω notation
- exps + logs

$I \subseteq \mathbb{R}$ with endpoints $a < b$ (possibly $a = -\infty$ and/or $b = \infty$). Let f be an integrable function on I ($f : I \to \mathbb{R}$)

Then, for any integers c, d such that $a \le c-1$ and $d+1 \le b$,

$$\int_{c-1}^{d} f(x)\,dx \le \sum_{i=c}^{d} f(i) \le \int_{c}^{d+1} f(x)\,dx$$

provided f is monotone ascending, and

$$\int_{c}^{d+1} f(x)\,dx \le \sum_{i=0}^{d} f(i) \le \int_{c-1}^{d} f(x)\,dx$$

if f is monotone descending.

Bounds on $\sum_{i=1}^{n} i^k$ (constant $k \ge 0$)

$f(i) := i^k$ is monotone ascending

So

$$\int_0^n x^k\,dx \le \sum_{i=1}^{n} i^k \le \int_1^{n+1} x^k\,dx$$

$$\left.\frac{1}{k+1}x^{k+1}\right|_0^n \qquad \left.\frac{1}{k+1}x^{k+1}\right|_1^{n+1}$$

$$\frac{n^{k+1}}{k+1} \le \sum_{i=1}^{n} i^k \le \frac{(n+1)^{k+1}-1}{k+1}$$

$$= \frac{n^{k+1}}{k+1} + O(n^k)$$

$$\therefore \Theta(n^{k+1})$$

[f monotone increasing]

$\sum_{i=c}^{d} f(i)$ is in an interval of length $\int_c^{d+1} f(x)\,dx - \int_{c-1}^{d} f(x)\,dx$

$$= \int_d^{d+1} f(x)\,dx - \int_{c-1}^{c} f(x)\,dx$$

[$f(x)$ on $[d, d+1]$ is $\le f(d+1)$

$f(x)$ on $[c-1, c]$ is $\ge f(c-1)$]

$$\le f(d+1) - f(c-1)$$

Quiz 1 answer:

$$\sum_{n=2}^{\infty} \frac{1}{n(\lg n)^2} < \infty$$

$$= \sum_{k=1}^{\infty} B_k \quad \text{where}$$

$$B_k = \frac{1}{2^k(\lg 2^k)^2} + \frac{1}{(2^k+1)(\lg(2^k+1))^2} + \dots + \frac{1}{(2^{k+1}-1)(\lg(2^{k+1}-1))^2}$$

$$\le \frac{1}{\cancel{2^k}(\lg 2^k)^2} \cdot \cancel{2^k}$$

$$= \frac{1}{(\lg 2^k)^2} = \frac{1}{k^2},$$

so $\sum_{k=1}^{\infty} B_k \le \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$ //

Seen O, Ω, Θ

next: o, ω

$f(n), g(n)$ functions (eventually positive)

$f(n) = o(g(n))$ means ....

Sep 1-12:52 PM

$f(n) = o(g(n))$ means

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

equivalently, $\forall c > 0$ $\exists n_0$ $\forall n \geq n_0$

$$f(n) \leq c\, g(n).$$

$f(n) = o(g(n))$ implies
$f(n) = O(g(n))$ and
$f(n) \neq \Omega(g(n))$

$n^2 = o(n^3)$

$\forall a < b$, $n^a = o(n^b)$

$\lg n = o(n^a)$ for $a > 0$.

$f(n) = \omega(g(n))$ means

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

equivalently, $g(n) = o(f(n))$

equivalently, $\forall c > 0$ $\exists n_0$ $\forall n \geq n_0$

$$f(n) \geq c\, g(n).$$

Exponentials:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

natural exponential.

If $a > 1$, define

$$a^x := e^{x \ln a}$$

$\ln x$ is natural log of x

$e^{\ln x} = x$ (for all $x > 0$)

$\ln(e^x) = x$ (for all real x)

$\lg$ abbrev for $\log_2$

$$\lg x = \frac{\ln x}{\ln 2}$$

More generally

$$\boxed{\log_b x = \frac{\log_a x}{\log_a b}}$$

for all $a, b > 0$, $\neq 1$.

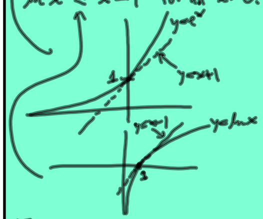
So

$$\log_a n = \Theta(\log_b n)$$

$(a, b > 1)$

$e^x \geq 1 + x$ for all $x \in \mathbb{R}$

$\ln x \leq x - 1$ for all $x > 0$.



For any $a > 1$, $k \geq 0$

$$n^k = o(a^n)$$

as $n \to \infty$.

$\ln n = o(n^k)$ for all $k > 0$

$\ln n = \omega(1)$

($\ln$ function is unbounded)

Sep 1-1:17 PM

<u>Thm</u>: $\displaystyle\sum_{p \text{ prime}} \frac{1}{p} = \infty.$

<u>Cor</u>: $\exists$ infinitely many primes.

<u>Proof</u>: Every positive integer n is the product of powers of distinct primes.

$150 = 2 \cdot 3 \cdot 5^2$

$$\infty = \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\leq \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right)\left(1 + \frac{1}{3} + \frac{1}{9} + \cdots\right)\left(1 + \frac{1}{5} + \frac{1}{25} + \cdots\right) \cdots \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right)$$

$$= \prod_{\text{prime } p} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right)$$

$$= \prod_{p} \left(\sum_{i=0}^{\infty} \left(\frac{1}{p}\right)^i \right)$$

$$= \prod_{p} \left(\frac{1}{1 - 1/p} \right)$$

$$= \prod_{p} \frac{p}{p-1} = \infty$$

Take $\ln$ of both sides

$$= \ln\left(\prod_{p} \frac{p}{p-1}\right) = \infty$$

$$= \sum_{p} \ln\left(\frac{p}{p-1}\right)$$

$$[\ln x \leq x - 1]$$

$$\leq \sum_{p} \left(\frac{p}{p-1} - 1\right)$$

$$= \sum_{p} \frac{1}{p-1} = \infty$$

$$\frac{1}{2-1} + \sum_{p>2} \frac{1}{p-1}$$

$$= 1 + \sum_{p>2} \frac{1}{p-1}$$

$$\leq 1 + \sum_{p} \frac{1}{p} = \infty$$

$\therefore \displaystyle\sum_{p} \frac{1}{p} = \infty.$ ∎

Sep 1-1:34 PM