Sums

$I \subseteq \mathbb{R}$ is an interval

$f : I \to \mathbb{R}$.

Def: f is monotone ascending on I

if $x \le y \Rightarrow f(x) \le f(y)$.

f is monotone descending on I.

if $x \le y \Rightarrow f(x) \ge f(y)$

$(\forall x, y \in I)$

f is monotone on I if either monotone asc. or monotone dec. on I.

$\underline{\text{lhs}} \le \underline{\text{rhs}}$

$\lg = \log_2$ monotone ascending on $(0, \infty)$

$\text{lhs} \le \text{rhs}$

$\Rightarrow \lg(\text{lhs}) \le \lg(\text{rhs})$

$f(x) = -x$ monotone dec.

$\text{lhs} \le \text{rhs}$

$\Rightarrow -\text{lhs} \ge -\text{rhs}$

$x \mapsto x^k \quad (k > 0)$ monotone asc $(0, \infty)$

$x \mapsto \frac{1}{x}$

Geometric Series. Fix $r \in \mathbb{R}$

$\sum_{k=0}^{n-1} r^k$ finite geometric series with ratio r.

$\sum_{k=0}^{\infty} r^k$ infinite " " " "

Case 1: $r = 1$: $\sum_{k=0}^{n-1} r^k = \sum_{k=0}^{n-1} 1 = n$

Case 2: $r \ne 1$:

$S := \sum_{k=0}^{n-1} r^k$

$rS := \sum_{k=0}^{n-1} r^{k+1} = \sum_{k=1}^{n} r^k$

$(1-r)S := \sum_{k=0}^{n-1} r^k - \sum_{k=1}^{n} r^k$

$= 1 - r^n$.

$$\boxed{S = \frac{1-r^n}{1-r}}$$

$\sum_{k=0}^{\infty} r^k = \lim_{n\to\infty} \sum_{k=0}^{n-1} r^k$

$= \frac{1}{1-r}$ (if $|r| < 1$)

E.g.,

$\sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{1-\frac{1}{2}} = 2$

$\sum_{k=0}^{\infty} k r^k \qquad (|r| < 1)$

$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$

$\frac{d}{dr} \sum_k r^k = \frac{d}{dr}\left(\frac{1}{1-r}\right)$

$\sum_{k=1}^{\infty} k r^{k-1} = \frac{d}{dr}(1-r)^{-1}$

$= (1-r)^{-2}(-1)(-1)$

$= \frac{1}{(1-r)^2}$

Multiply both sides by r:

$\sum_{k=0}^{\infty} k r^k = \sum_{k=1}^{\infty} k r^k = \frac{r}{(1-r)^2}$

$r = \frac{1}{2}$: $\sum_{k=0}^{\infty} k 2^{-k} = \frac{1/2}{(1-\frac{1}{2})^2} = 2$

expected # of heads before tails

$\Pr[k \text{ heads}] = \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)$

$\sum_{k=0}^{\infty} k \Pr[k \text{ heads}]$

$= \frac{1}{2} \sum_{k=0}^{\infty} k 2^{-k} = 1$.

$\sum_{k=0}^{\infty} k^2 r^k$ Take prev closed form for $\sum k r^k$, differentiate by r, multiply by r.

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Bounding sums asymptotically

$f(n) = O(g(n))$ last time

f, g are asymptotically positive
($\exists n_0\ \forall n \geq n_0\ f(n) > 0\ \&\ g(n) > 0$)

$f(n) = \Omega(g(n))$

means $\exists c > 0\ \exists n_0\ \forall n \geq n_0$
$f(n) \geq c\, g(n)$.

(equivalently, $g(n) = O(f(n))$)

("g grows no faster than f asymptotically").

$f(n) = \Theta(g(n))$ means
$f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

("f & g have the same asymptotic growth rate.")

equivalently: $\exists\ 0 < c_1 \leq c_2\ \exists n_0$
$\forall n \geq n_0$
$\underbrace{c_1 g(n) \leq f(n) \leq c_2 g(n)}$.

Apply to sums.

$$f(n) = \sum_{k=0}^{n} k^7 = \Theta(n^8)$$

Tight asymptotic bound on $f(n)$

~~f(k)~~ k^7 is monotone ascending in k

$$f(n) = \sum_{k=0}^{n} k^7 \leq \sum_{k=1}^{n} n^7 = n \cdot n^7 = n^8$$

$\therefore f(n) = O(n^8)$ (upper bound)

Lower bound:

$$f(n) = \sum_{k=1}^{n} k^7 \geq \sum_{k=\lceil \frac{n}{2} \rceil}^{n} k^7$$

smallest term:

$$\geq \sum_{k=\lceil \frac{n}{2} \rceil}^{n} \lceil \tfrac{n}{2} \rceil^7 \geq \sum_{k=\lceil \frac{n}{2} \rceil}^{n} \left(\tfrac{n}{2}\right)^7$$

$$\geq \left(\tfrac{n}{2}\right)\left(\tfrac{n}{2}\right)^7 = \left(\tfrac{n}{2}\right)^8 = \frac{1}{2^8} n^8$$

$\therefore f(n) = \Omega(n^8)$

"splitting the sum"

For any monotone ascending $a(k)$ such that

$a(\frac{k}{2}) \geq c\, a(k)$ ($a(k) = k^7$)

$$\frac{n\, a(n)}{2c} \leq \sum_{k=1}^{n} a(k) \leq n\, a(n)$$

$$= \Theta(n\, a(n))$$

Ex: $\sum_{k=1}^{n} \lg k \leq n \lg n$

$$\frac{n \lg(\frac{n}{2})}{2} = \frac{n(\lg n - \lg 2)}{2}$$

$$= \frac{n \lg n}{2} - \frac{n}{2}$$

$$\geq \frac{n \lg n}{3}$$ for large enough n.

Thm: For any $a \geq 0$,

arithmetic sum
$$\sum_{k=1}^{n} k^a = \Theta(n^{a+1})$$

Pf: Use the technique.

Harmonic Sum

$$\sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{n} k^{-1}$$

$$= \Theta(\lg n)$$

Given n, pick a integer such that $2^a \leq n < 2^{a+1}$

$a = \lfloor \lg n \rfloor$

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$$\sum_{k=1}^{n} = \underbrace{1}_{B_0} + \underbrace{\frac{1}{2} + \frac{1}{3}}_{B_1} + \underbrace{\frac{1}{4} + \frac{1}{5} \cdots}_{B_2}$$

~~next~~ 4 terms

Each block B_i has 2^i terms

~~Upper bo~~

$$B_i = \frac{1}{2^i} + \frac{1}{2^i+1} + \cdots + \frac{1}{2^{i+1}-1}$$

$$\frac{1}{2^{i+1}} + \frac{1}{2^{i+1}} + \cdots \frac{1}{2^{i+1}} \le B_i \le \frac{1}{2^i} + \frac{1}{2^i} + \cdots + \frac{1}{2^i}$$

$$\le 1$$

$$= \frac{2^i}{2^{i+1}} = \frac{1}{2}$$

$$\boxed{\frac{1}{2} \le B_i \le 1}$$

$$\sum_{k=1}^{n} \frac{1}{k} = \sum_{j=0}^{a-1} B_j \qquad a = \lfloor \lg n \rfloor$$

$$+ \underbrace{\frac{1}{2^a} + \frac{1}{2^a+1} + \cdots \frac{1}{n}}_{\le B_a}$$

$$\sum_{j=0}^{a-1} B_j \le \sum_{k=1}^{n} \frac{1}{k} \le \sum_{j=0}^{a} B_j$$

$$\sum_{j=0}^{a-1} B_j \ge \sum_{j=0}^{a-1} \frac{1}{2} = \frac{a}{2} = \frac{\lfloor \lg n \rfloor}{2}$$

$$\sum_{j=0}^{a} B_j \le \sum_{j=0}^{a} 1 \le a+1 = \lfloor \lg n \rfloor + 1$$

$$\sum_{k=1}^{n} \frac{1}{k} = \Theta(\lg n)$$

Products

$$f(n) = \prod_{k=0}^{n} 2^k = 2^{\sum_{k=0}^{n} k}$$

Fallback method.

$$\lg f(n) = \sum_{k=0}^{n} \lg(2^k) = \sum_{k=0}^{n} k$$

$$= \frac{n(n+1)}{2}$$

$$f(n) = 2^{\lg f(n)} = 2^{\frac{n(n+1)}{2}}$$

Method: take the lg
turns product into a sum
take 2 to the power of
the result.

Review facts about logarithms

Base $b > 0$ $(b \neq 1)$. $a > 0$

$\log_b a$ = that unique x such that $b^x = a$

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b(x^r) = r \log_b x$$

$$a^{\log_b c} = c^{\log_b a}$$

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