CSCE 750, Fall 2009 — Quizzes with Answers

Stephen A. Fenner

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1. Give an exact closed form for
$$\sum_{k=1}^{\infty} \frac{2k}{3^{k+1}}.$$
Simplify your answer as much as possible.

   Answer: We reduce the expression to a form we've already seen in class:
$$\sum_{k=1}^{\infty} \frac{2k}{3^{k+1}} = \frac{2}{3}\sum_{k=1}^{\infty} \frac{k}{3^k} = \frac{2}{3}\sum_{k=1}^{\infty} kr^k \ ,$$
where $r = 1/3$. We saw in class[1] that the sum on the right is
$$\sum_{k=1}^{\infty} kr^k = \frac{r}{(1-r)^2}$$
for all r such that $|r| < 1$. Thus the final answer is
$$\frac{2}{3} \cdot \frac{1/3}{(1-1/3)^2} = \frac{2}{3} \cdot \frac{1/3}{4/9} = \frac{1}{2} \ .$$

2. Let f a real-valued function defined on $\mathbb{R}$.

   Recall:

   - $f(n)$ is *strictly monotone increasing* iff
   $$x < y \implies f(x) < f(y)$$
   for all $x, y \in \mathbb{R}$.
   - $f(n)$ is *strictly monotone decreasing* iff
   $$x < y \implies f(x) > f(y)$$
   for all $x, y \in \mathbb{R}$.

[1] If you forgot the formula, re-derive it as follows: (1) start with the formula for an infinite geometric series, $\sum_{k=0}^{\infty} r^k = 1/(1-r)$; (2) differentiate both sides with respect to r; (3) multiply both sides by r.

Show that if $f(n)$ and $g(n)$ are both strictly monotone *decreasing*, then $f(g(n))$ is strictly monotone *increasing*.

Answer: For all $x, y \in \mathbb{R}$, we have

$$\begin{aligned} x < y &\implies g(x) > g(y) && (g \text{ is strictly decreasing}) \\ &\implies f(g(x)) < f(g(y)) \ . && (f \text{ is strictly decreasing}) \end{aligned}$$

Thus $f(g(n))$ is strictly increasing.

3. Use the substitution method to show that if

$$T(n) = 4T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n^2,$$

then

$$T = O(n^2 \lg n).$$

Only show the inductive step. Don't worry about any base case(s).

Answer: We tacitly assume that $T(n)$ is eventually positive. Fix n large enough, and assume (inductive hypothesis) that $T(m) \le Cm^2 \lg m$ for all $m < n$, where C is a constant to be chosen later. Then

$$\begin{aligned} T(n) &= 4T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n^2 \\ &\le 4C\left\lfloor \frac{n}{2} \right\rfloor^2 \lg \left\lfloor \frac{n}{2} \right\rfloor + n^2 && \text{(inductive hypothesis)} \\ &\le 4C\left(\frac{n}{2}\right)^2 \lg \frac{n}{2} + n^2 && \text{(monotonicity)} \\ &= Cn^2(\lg n - 1) + n^2 \\ &= Cn^2 \lg n + (1 - C)n^2 \\ &\le Cn^2 \lg n \ , \end{aligned}$$

provide $(1 - C)n^2 \le 0$, that is, provided $C \ge 1$. Setting $C := 1$ then suffices for the induction.

4. Prove the identity

$$\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}$$

for integers $0 < k \le n$.

Answer:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n-k+1}{k} \cdot \frac{n!}{(k-1)!(n-k+1)!} = \frac{n-k+1}{k}\binom{n}{k-1} \ .$$

5. Show how to generate an integer from 1 to 5 uniformly at random by rolling a regular 6-sided die. (Your procedure should terminate with probability 1).

   What is the expected number of die rolls you need?

   [Hint: if you did it correctly, the number of die rolls should be geometrically distributed.]

   Answer: Roll the die repeatedly until a number in the range $1, \ldots, 5$ appears. That is, roll the die until some number besides 6 appears. Take the final number as the output. The number N of rolls is geometrically distributed with success probability $p = 5/6$, which means that the probability of exactly k rolls is $\Pr[N = k] = p(1-p)^{k-1}$, for $k = 1, 2, 3, \ldots$. Thus

$$E[N] = \sum_{k=1}^{\infty} k \cdot \Pr[N = k] = \frac{p}{1-p} \sum_{k=1}^{\infty} k(1-p)^k = \frac{p}{1-p} \cdot \frac{1-p}{p^2} = \frac{1}{p} = \frac{6}{5} .$$

6. Consider the (binary) min heap

$$A = \langle 5, 7, 11, 15, 8, 12, 13, 16, 20, 14, 18, 19 \rangle .$$

   (Indices start at 1.)

   (a) Give the result of the operation Heap-Extract-Min (i.e., DeleteMin) on A.

   (b) Give the result of the operation Min-Heap-Insert$(A, 10)$.

   Give both answers as lists of numbers (not trees). Do not do the operations in sequence; each operation is applied to the original unaltered A.

   Answer:

   (a) $\langle 7, 8, 11, 15, 14, 12, 13, 16, 20, 19, 18 \rangle$

   (b) $\langle 5, 7, 10, 15, 8, 11, 13, 16, 20, 14, 18, 19, 12 \rangle$

7. Describe a $\Theta(n)$-time algorithm that takes as input

   - an unsorted array $A[1 \ldots n]$ of n distinct numbers and
   - an integer k with $1 \leq k \leq n$

   and returns an array $B[1 \ldots k]$ of the k smallest numbers from the list $A[1 \ldots n]$. The array B need not be sorted.

   You may assume that a linear-time Selection algorithm is available as a subroutine.

   Either pseudocode or an informal description will suffice.

   Answer: First use the linear-time Selection algorithm to find the k-th smallest number in A. Call this number x. Then scan linearly through the array A once, copying $A[i]$ into B iff $A[i] \leq x$. (Note that the numbers in A are assumed to be distinct, so ties are not a problem.)

8. Explain why the longest simple path from a node x in a red-black tree to a descendant leaf has length at most twice that of the shortest simple path from node x to a descendant leaf.

   Answer: All paths down from x have the same number b of black nodes. Since these paths all end with a black node, and there are no two red nodes in a row, the number r of red nodes along any path from x satisfies $r \leq b$. So the length $r + b$ of any such path is at least b and at most $2b$.

9. Give the entries in the m-table (stealth bomber table) in the matrix chain order computation for A_1, A_2, A_3, A_4 with sequence of dimensions

   $$\langle p_0, p_1, p_2, p_3, p_4 \rangle = \langle 4, 6, 5, 2, 3 \rangle.$$

   You do not need to show any side calculations for full credit.

   5 points EXTRA CREDIT: Give the s-table and the optimal parenthesization.

   [Note: in class, I named the m-table table the c-table with different indices. You do not need to show indices, so the correct answer is the same regardless.]

   Answer: The optimal parenthesization is $(A_1(A_2A_3))A_4$, with a total of 132 multiplications. Here is the table, rotated 45° clockwise:

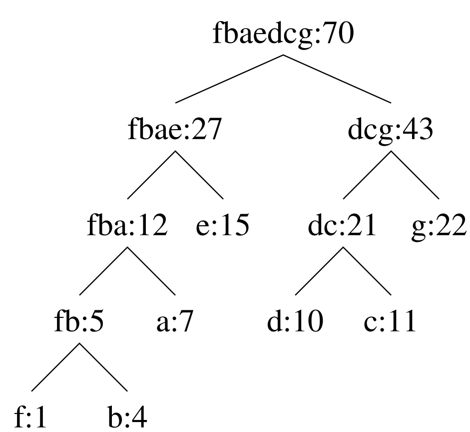
0	120	108	132
1	0	60	96
1	2	0	30
3	3	3	0

10. Draw an optimal Huffman encoding tree for the following set of chars and frequencies:

    `a:7   b:4   c:11   d:10   e:15   f:1   g:22`

    Your tree should be the same as that produced by the algorithm described in class or in the book, i.e., a left child should always have frequency less than or equal to that of its right sibling.

    Answer:



11. Prof. Pangloss suggests an alternate potential function for the binary counter: the **number of rightmost consecutive 1's** (starting with the l.s.b. and going left to the rightmost 0). He says (correctly) that this function adequately pays for all carry operations.

    Show that this potential does *not* give constant amortized time for all increment operations.

    [Hint: How does this potential change when incrementing from 62 to 63?]

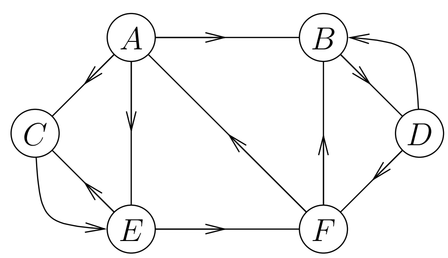
    Answer: When incrementing from $2^k - 2$ to $2^k - 1$ for some k, e.g., from 62 to 63, the potential function rises from 0 to k:

$$\cdots 00\underbrace{11\cdots 1}_{k-1}0 = 2^k - 2 \qquad \text{(potential is 0)}$$

$$\cdots 00\underbrace{11\cdots 1}_{k-1}1 = 2^k - 1 \qquad \text{(potential is } k\text{)}$$

    and so the amortized time of this increment is $k + 1$, which is not constant, because k is unbounded.

12. Give the d- and π-values for each node after running breadth-first search on the graph below, starting at D:

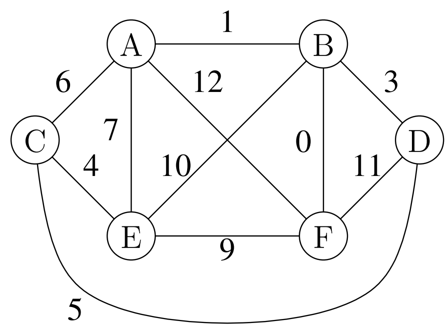


    Give the values in tabular form, sorted alphabetically by vertex label.

    Answer:

v	$v.d$	$v.\pi$
A	2	F
B	1	D
C	3	A
D	0	NIL
E	3	A
F	1	D

13. Give the π-value of each node (except D) after it is removed from the priority queue when running Prim's algorithm on the graph below, starting at D:



Give the values in tabular form, in chronological order by queue removal.

Answer:

v	$v.\pi$
D	NIL
B	D
F	B
A	B
C	D
E	C