csce750 — Analysis of Algorithms
Fall 2020 — Lecture Notes: Solving Recurrences

This document contains slides from the lecture, formatted to be suitable for printing or individual reading, and with some supplemental explanations added. It is intended as a supplement to, rather than a replacement for, the lectures themselves — you should not expect the notes to be self-contained or complete on their own.

1 Definition

CLRS 4.3–4.5

A **recurrence** is an equation or inequality that describes a function in terms of its own value on smaller inputs.

We've already seen one example (for the run time of MergeSort):

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq 1 \\ 2T(\frac{n}{2}) + \Theta(n) & \text{otherwise} \end{cases}$$

Recurrences are important because they are the primary tool for analyzing recursive algorithms.

We'll look at three different ways to solve recurrences.

- Substitution method
- Recursion trees
- The Master theorem

2 Smoothness rule

Definition A non-negative function $f(n)$ is called **smooth** if

$$f(2n) \in \Theta(f(n)).$$

Example: $f(n) = n^3$ is a smooth function, because

$$f(2n) = (2n)^3 = 8n^3 = \Theta(n^3).$$

Example: $g(n) = 2^n$ is not a smooth function, because

$$g(2n) = 2^{2n} \neq \Theta(2^n).$$

Smoothness rule (informally): Suppose we show, for some $b \geq 2$, that $f(n) = \Theta(g(n))$ when n is a power of b. Then, if $f(n)$ is a smooth function, we have $f(n) = \Theta(g(n))$ for *all* n.

Smoothness rule (even more informally): Most of the time, floors and ceilings do not affect the asymptotic growth rate.

3 Substitution method

The **substitution method** for solving recurrences has two parts.

1. **Guess** the correct answer.
2. **Prove** by induction that your guess is correct.

4 Example

Use the substitution method to solve $T(n) = 2T(n/2) + n$.

Guess: $T(n) = O(n \lg n)$

Proof: Use induction on n to show that there exist c and n_0 for which $T(n) \le cn \lg n$ for all $n \ge n_0$.

- Base case: Almost always omitted, because $T(n) = \Theta(1)$ when n is sufficiently small, so we can always choose c large enough.

  Details: CLRS 84

- Induction step: Assume that $T(m) \le cm \lg m$ for all $m < n$, to prove that $T(n) \le cn \lg n$.

  (We'll find restrictions on c and n_0 along the way.)

$$\begin{aligned} T(n) &= 2T(n/2) + n \\ &\le 2 \cdot c(n/2)\lg(n/2) + n \\ &= cn\lg(n/2) + n \\ &= cn\lg n - cn\lg 2 + n \\ &= cn\lg n - cn + n \\ &\le cn\lg n \qquad [\text{ when } c \ge 1] \end{aligned}$$

In the last step, we replace $-cn + n$ *with* 0*. This increases the sum if*

$$-cn + n \le 0.$$

Solve for c *to get the constraint* $c \ge 1$*.*

5 Be careful!

Substitution proofs must ensure that they use the **same constant** as in the inductive hypothesis.

Here's an example of how to "prove" (incorrectly!) that $T(n) = O(n)$.

$$\begin{aligned} T(n) &= 2T(n/2) + n \\ &\le 2c(n/2) + n \\ &= (c+1)n \\ &= O(n) \end{aligned}$$

The problem is that we have not proved the exact form of the inductive hypothesis. *In particular, the constant c we use when we substitute at the beginning must be the same c we have at the end of the inequalities.*

6 Even correct guesses can lead to dead ends

Show that $T(n) = 5T(n/2) + n^2$ is $O(n^{\log_2 5})$.

Attempt 1: Use induction to (try to) show that, for some c,

$$T(n) \leq cn^{\log_2 5}.$$

"Proof:"

$$\begin{aligned} T(n) &= 5T(n/2) + n^2 \\ &\leq 5\left(c\left(\frac{n}{2}\right)^{\log_2 5}\right) + n^2 \\ &= 5\left(c\frac{n^{\log_2 5}}{2^{\log_2 5}}\right) + n^2 \\ &= cn^{\log_2 5} + n^2 \end{aligned}$$

A dead end! No choice of c makes this inequality true.

7 Proving a stronger bound

Show that $T(n) = 5T(n/2) + n^2$ is $O(n^{\log_2 5})$.

Attempt 2: Use induction to show that, for some c and some a,

$$T(n) \leq cn^{\log_2 5} - an^2.$$

Note that if we can show that $T(n) \leq cn^{\log_2 5} - an^2$ for some positive constant a, then we know immediately that $T(n) \leq cn^{\log_2 5}$, which is sufficient to show that $T(n) \in O(n^{\log_2 5})$. So this strong bound really is doing the job that we need it to.

The choice of adding $-an^2$ here is based on the dead end from the previous slide — we had an extra n^2 term, and we're hoping that the new term in the inductive hypothesis will counteract that. This doesn't always work out, but it often does, and it's a good place to start.

Proof:

$$\begin{aligned} T(n) &= 5T(n/2) + n^2 \\ &\leq 5\left(c\left(\frac{n}{2}\right)^{\log_2 5} - a\left(\frac{n}{2}\right)^2\right) + n^2 \\ &= 5\left(c\frac{n^{\log_2 5}}{2^{\log_2 5}} - a\frac{n^2}{4}\right) + n^2 \\ &= cn^{\log_2 5} + \left(1 - \frac{5a}{4}\right)n^2 \\ &\leq cn^{\log_2 5} - an^2 \qquad [a \geq 4] \end{aligned}$$

This is enough to conclude $T(n) = O(n^{\log_2 5})$, because $cn^{\log_2 5} - 4n^2 \leq cn^{\log_2 5}$.

In the last step of the proof, we replace $\left(1 - \frac{5a}{4}\right)$ with $-a$. We want $-a$, because we need to match the exact form of the inductive hypothesis.

This replacement increases the expression (or leaves it unchanged), making the $\leq$ we write there correct, when

$$\left(1 - \frac{5a}{4}\right) \leq -a.$$

Solving for a (first multiply both sides by 4, then add $5a$ to both sides) we get:

$$\begin{aligned} 4 - 5a &\leq -4a \\ 4 &\leq a \end{aligned}$$

Hence the constraint $a \geq 4$.

8 Change of variables

Solve the recurrence $T(n) = 2T(\sqrt{n}) + \lg n$.

Solution: Change of variables. Let $m = \lg n$ and $S(m) = T(2^m)$.

Note that $n = 2^m$ and $\sqrt{n} = 2^{m/2}$.

Then we get:

$$\begin{aligned} T(n) &= T(2^m) \\ &= 2T(2^{m/2}) + \lg(2^m) \\ &= 2S(m/2) + m \\ &= O(m \log m) \\ &= O(\log n \log\log n) \end{aligned}$$

Recall that we've solved the recurrence $S(m) = 2S(m/2)+n$ already, a few slides back.

9 Recursion trees

We can solve many recurrences by drawing a **recursion tree**.

- **Nodes**: Label with the contribution to the total for that 'recursive call'.

  ...**not** counting what happens inside children.

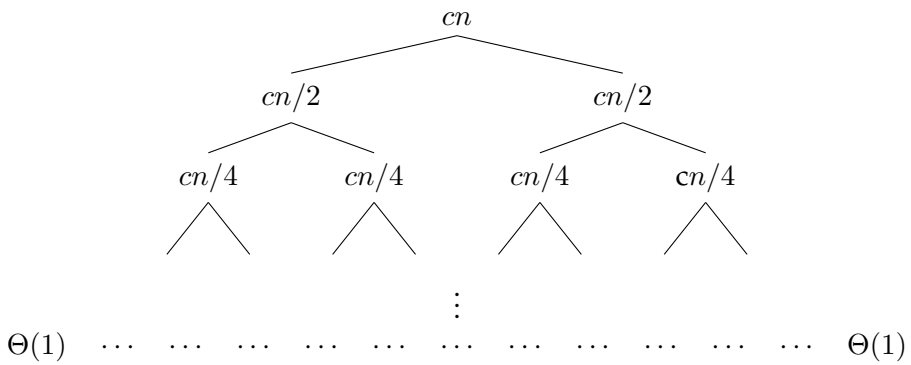
- **Children**: One for each appearance of a recurrent term.

After drawing such a tree, we can solve the recurrence:

1. Compute (or bound) the **depth of the leaves**.
2. Compute (or bound) the **sum for each level**.
3. Compute (or bound) the **sum across all levels**.

10 Example: Mergesort recurrence

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq 1 \\ 2T(\frac{n}{2}) + \Theta(n) & \text{otherwise} \end{cases}$$

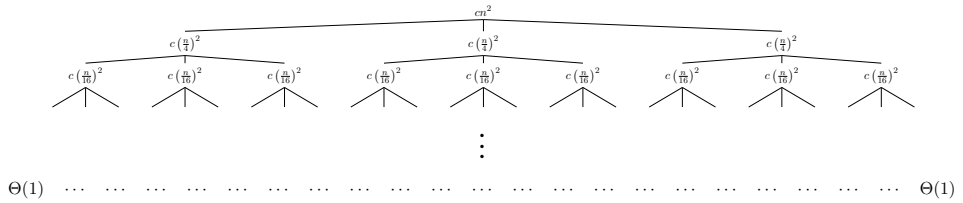


11 Example: Mergesort recurrence

- Depth of the leaves: $\lg n$
- Sum for each level: cn
- Sum across all levels: $cn \lg n = \Theta(n \log n)$.

12 Example: Another divide-and-conquer recurrence

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \le 1 \\ 3T(\frac{n}{4}) + \Theta(n^2) & \text{otherwise} \end{cases}$$



13 Example continued

- Depth of the leaves: $\log_4 n$
- Sum for each level: $3^i c(n/4^i)^2 = (3/16)^i cn^2$.
- Sum across all levels:

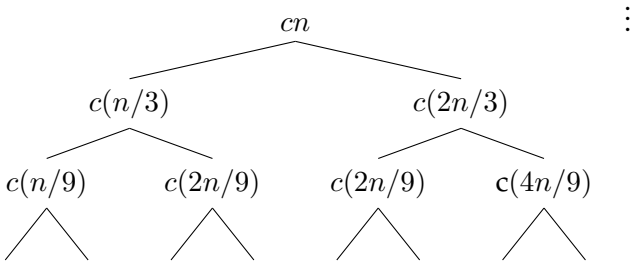
$$\begin{aligned} T(n) &= \sum_{i=0}^{\log_4 n} \left(\frac{3}{16}\right)^i cn^2 \\ &\le \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i cn^2 \\ &= \frac{1}{1-(3/16)} cn^2 \\ &= \frac{16}{13} cn^2 \\ &= O(n^2) \end{aligned}$$

Note also: $T(n) = \Omega(n^2)$. (Why?)

> *At depth i, the 'problem size' is $n/2^i$. To get down to to the base case, we need this value to be 1 or less.*

14 A lopsided tree

$T(n) = T(n/3) + T(2n/3) + O(n)$

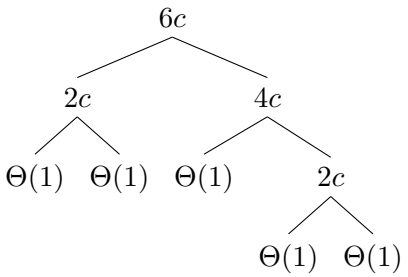


- Depth of the (deepest) leaves: $\log_{3/2} n$
- Sum for each level: $\leq cn$
- Sum across all levels: $cn \log_{3/2} n = O(n \log n)$.

15 Some branches terminate before others

Note that this recurrence does not produce a complete tree!

For $n = 6$ (assuming $\lfloor n/3 \rfloor$ and $\lfloor 2n/3 \rfloor$):



Therefore, the sum from the previous slide gives an upper bound. We could also get a lower bound by truncating the tree at the level of its shallowest leaves.

16 Master theorem: Simple version

Theorem: Consider the recurrence

$$T(n) = aT(n/b) + \Theta(n^d).$$

If $a > b^d$ then $T(n) = \Theta(n^{\log_b a})$.
If $a = b^d$ then $T(n) = \Theta(n^d \log n)$.
If $a < b^d$ then $T(n) = \Theta(n^d)$.

For this simple version, the final added part must be a polynomial.

17 Master theorem: Real version

Theorem: Consider the recurrence

$$T(n) = aT(n/b) + f(n).$$

1. If there exists $\epsilon > 0$, for which $f(n) = O(n^{\log_b a - \epsilon})$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$.
3. If there exists $\epsilon > 0$, for which $f(n) = \Omega(n^{\log_b a + \epsilon})$, and $af(n/b) \leq cf(n)$ for some constant c and sufficiently large n, then $T(n) = \Theta(f(n))$.

18 Example 1

$$T(n) = 9T(n/3) + n$$

We have $a = 9$, $b = 3$, and $f(n) = n$.

Compare $n^{\log_3 9} = n^2$ to n. Observe that $n = O(n^{2-\epsilon})$, with $\epsilon = 1$.

Therefore, the first case applies, and $T(n) = \Theta(n^{\log_b a}) = \Theta(n^2)$.

19 Example 2

$$T(n) = T(2n/3) + 1$$

We have $a = 1$, $b = 3/2$, and $f(n) = 1$.

Compare $n^{\log_{3/2} 1} = 1$ to 1. Observe that $1 = \Theta(1)$.

Therefore, the second case applies, and $T(n) = \Theta(n^{\log_b a} \log n) = \Theta(\log n)$.

20 Example 3

$$T(n) = 3T(n/4) + n \log n$$

We have $a = 3$, $b = 4$, and $f(n) = n \log n$.

Compare $n^{\log_4 3}$ to $n \log n$. Observe that $n \log n = \Omega(n^{\log_4 3+\epsilon})$, as long as $\log_4 3 + \epsilon \leq 1$. (For example, choose $\epsilon = 0.2$.)

The "regularity condition" also holds.

Therefore, the third case applies, and $T(n) = \Theta(n \log n)$.

21 Example 4

$$T(n) = 2T(n/2) + n \log n$$

We have $a = 2$, $b = 2$, and $f(n) = n \log n$.

Compare $n^{\log_2 2} = n$ to $n \log n$. Observe that, although $n \log n = \Omega(n)$, for any $\epsilon > 0$, $n \log n \neq \Omega(n^{1+\epsilon})$.

(Intuition: Even for a very small ϵ, $n^{1+\epsilon}$ will eventually grow faster than $n \log n$.)

Therefore, the Master theorem does not apply.