✓TQBF is NPSPACE-complete
✓Savitch's Theorem & corollary
Generalized Geography (GG)

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<u>Thm</u>: TQBF is NPSPACE-complete

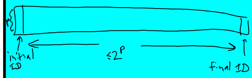
<u>Proof</u>: Given $L \in$ NPSPACE, Let M be a PSPACE TM (with help tape) that decides L.

Let $p(n)$ be a polynomial big enough so that an ID of M has $\leq p(n)$ bits for any input $w \in \Sigma^n$, and M runs in time $\leq 2^{p(n)}$ on such inputs.

Fix input w of length n, and set $p := p(n)$.

By <u>ID</u> we reprepresent a string of p many Boolean variables [(encoding an ID)]

M on input w's tableau:




For $0 \leq t \leq p$, let

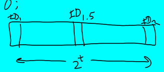
$\varphi_t(ID_1, ID_2)$ = "$ID_1$ & $ID_2$ are legal IDs of M and $ID_2$ is reachable from $ID_1$ in $\leq 2^t$ many steps (via the guess tape)

If $t=0$:

$\varphi_0(ID_1, ID_2)$ = Either $ID_1 = ID_2$ or M can go from $ID_1$ to $ID_2$ in one step, reading one symbol on the guess tape (actually, there exist such a symbol).

(Details similar to the Cook-Levin Thm proof).

$t > 0$:




$$\varphi_t(ID_1, ID_2) \equiv \exists ID_{1.5}\Big(\varphi_{t-1}(ID_1, ID_{1.5}) \wedge \varphi_{t-1}(ID_{1.5}, ID_2)\Big)$$

$$\equiv \exists ID_{1.5}\, \forall x\, \exists \vec{y}, \vec{z} \Big[ \big[(x=0 \wedge \vec{y} = ID_1 \wedge \vec{z} = ID_{1.5}) \vee (x=1 \wedge \vec{y} = ID_{1.5} \wedge \vec{z} = ID_2)\big] \wedge \underbrace{\varphi_{t-1}(\vec{y}, \vec{z})} \Big]$$

Then $\varphi_p$ has size $O(p^2)$.

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p-reduction $L \leq_p$ TQBF:

"On input w: // w has length n

1. Let $F :=$
$(\exists ID_0, ID_a)\Big[ (ID_0$ is the initial ID of M on input w$)$ $\wedge$ $(ID_a$ is an accepting ID of M$)$ $\wedge\ \varphi_p(ID_0, ID_a) \Big]$
2. Output F." ▨

<u>Thm (Savitch)</u>: For any "reasonable" space bound $s(n)$,

$$\text{NSPACE}(s(n)) \subseteq \text{SPACE}(s(n)^2).$$

<u>Proof Idea</u>: Given $L \in \text{NSPACE}(s(n))$, on w, $|w| = n$, construct F as in the previous proof.

F has size $O(s(n)^2)$

Computing F takes space $O(|F|)$, so "$w \in L$" can be decided in (deterministic space $O(s(n)^2)$.

<u>Corr</u>: NPSPACE = PSPACE.

Prop: $ALL_{NFA} \in PSPACE$

Proof Idea: Given an NFA N, guess an input string that N rejects.

[guess means read the guess tape]:

simulate N on the symbols on the guess tape.

This shows that

$\overline{ALL_{NFA}} \in NPSPACE$

$\therefore \overline{ALL_{NFA}} \in PSPACE$ (by th Cor)

$\therefore ALL_{NFA} \in PSPACE$

(PSPACE is closed under complement)

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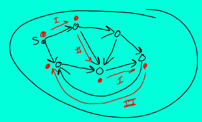
Generalized Geography (GG)

Digraph G, and start vertex $s \in G.V$. Two players alternating turns: a turn is moving the current vertex along a directed edge to a new vertex. Initially, the current vertex is s.




First player unable to move to a vertex not visited before loses.

GG:

Instance: A graph G and a vertex $s \in G.V$.

Question: Does Player I have a winning strategy in the GG game starting at s?

Prop: $TQBF \leq_p GG$.

Proof: By construction.

Let F be be any q.b.f.

Claim: We can assume that

$F = (\exists x_1)(\forall x_2) \cdots (\exists x_n)\varphi(x_1, \ldots, x_n)$

where n is odd & quantifiers alternate, starting with $\exists$ & $\varphi$ is in cnf.

[ $\cdots (\exists x)\ (\exists y) \cdots$

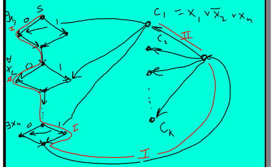
$(\forall v)$

v occurs nowhere else ]

Given F, where

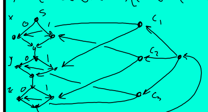
$\varphi = C_1 \wedge \cdots \wedge C_k$ (cnf)

Build (G, s) as follows:




Example:

$F = (\exists x)(\forall y)(\exists z)[(x \vee \overline{y}) \wedge (\overline{x} \vee \overline{z}) \wedge (y \vee z)]$

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Is F true: