Prop: PSPACE is closed under $\leq_p$ ($A \leq_p B$ & $B \in$ PSPACE $\Longrightarrow$ A is in PSPACE).

Proof (similar to the case for P and NP).

Def: Let φ be a Bool fmla, x a variable in φ, and $b \in \{0,1\}$. Let $\varphi[x:=b]$ be the Bool fmla obtained by substituting the constant b in for x (all occurrences of x).

Def: A quantified Boolean formula (qbf) is of the form

$$(Q_1 x_1)(Q_2 x_2) \cdots (Q_n x_n)\varphi,$$

where φ is a Bool fmla with variables $x_1, \ldots, x_n$ (no others), and each Q_i is either $\exists$ or $\forall$.

$(\exists x)\varphi$ means φ is true for at least one of the truth values of x ($\varphi[x:=0]$ or $\varphi[x:=1]$)

$(\forall x)\varphi$ means φ is true for both truth values of x ($\varphi[x:=0]$ and $\varphi[x:=1]$)

Ex:

$$(\exists x)(\forall y)(\exists z)\left[(x \wedge y \wedge z) \vee (x \wedge \bar{y} \wedge \bar{z})\right]$$

true

$$\text{- - - - - - - - -}\left[(x \wedge y \wedge z) \vee (\bar{x} \wedge \bar{y} \wedge \bar{z})\right]$$

false

Quantifier order matters!

Def: TQBF (True Quantified Boolean Formulas) is the decision problem:

Instance: A qbf F

Question: Is F true?

Theorem: TQBF is PSPACE-complete (under p-reducibility), that is

1) TQBF $\in$ PSPACE
2) For every $L \in$ PSPACE, $L \leq_p$ TQBF.

Proof:

(1)

D := "On input F, a qbf:

1. If F has no variables, then accept if F=1 and reject otherwise
2. Otherwise, $F = (Qx)G$ for some fmla G
   a) If $Q = \exists$, then accept iff $D(G[x:=0])$ accepts or $D(G[x:=1])$ accepts.
   b) If $Q = \forall$, then accept iff $D(G[x:=0])$ accepts and $D(G[x:=1])$ accepts."

polyspace because control stack contains at most n records at any given time (n = #vars in F).

(2) $L \in$ PSPACE, input x