$3\text{-SAT} \leq_p \overline{ALL_{NFA}}$

$\varphi \longrightarrow N$

Given $\varphi := C_1 \wedge \cdots \wedge C_k$

variables $x_1, \ldots, x_n$

Construct $N = \langle Q, \Sigma, \delta, q_0, F \rangle$

where $\Sigma = \{0, 1\}$

$Q := \{q_0\} \cup \{q_{i,j} : 1 \leq i \leq n+1, 1 \leq j \leq k\}$

$F := Q$

$\delta(q_0, \varepsilon) = \{q_{1,j} : 1 \leq j \leq k\}$

// j indexes the clause C_j

// i indexes the variable x_i

$\forall i, 1 \leq i \leq n, \forall j, 1 \leq j \leq k,$

$\delta(q_{i,j}, b)$

$= \{q_{i+1,j}\}$ if either x_i or $\overline{x_i}$ does not occur in C_j, or literal x_i occurs positively in C_j and $b=0$ or $\overline{x_i}$ occurs in C_j and $b=1$.

otherwise $\delta(q_{i,j}, b) = \emptyset$

and $\forall b, \forall j,$

$\delta(q_{n+1,j}, b) = \{q_{n+1,j}\}$

N accepts all binary strings except those that have a sat assignment to φ as a prefix.

$\therefore \langle N \rangle \in \overline{ALL_{NFA}}$ iff φ is satisfiable ($\varphi \in 3\text{-SAT}$). //

Space-bounded computation.

Let M be a TM.

On input w, the <u>space</u> used by M on input w is the number of tape cells scanned at least once.

For any function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) \geq n$ ($\forall n$)

$SPACE(f(n))$ is the class of langs <u>decided</u> by TMs that use space $\leq O(f(n))$ on all inputs of length n ($\forall n$).

$$PSPACE = \bigcup_{k=1}^{\infty} SPACE(n^k)$$

$$= SPACE(Poly(n))$$

<u>Easy</u>: $P \subseteq PSPACE$.

Reason: Can only scan a contant # of new cells per time step (constant = # of tapes).

<u>Prop</u>: $PSPACE \subseteq \underbrace{DTIME(2^{poly(n)})}_{EXP}$

<u>Proof</u>: <u>Idea</u>: A TM using space $S \in \mathbb{N}$ can only have $2^{O(S)}$ many IDs. Why?

# of possible IDs is

$|\Gamma|^s$ = # possible tape contents

$\times$

$|Q|$ = # of possible states

$\times$

s = # of possible head positions

$= |Q| \cdot s \cdot |\Gamma|^s$

$= O(s|\Gamma|^s)$

$= O(\underline{sc^s})$ (some const c)

$= O(\overline{s2^{s \lg c}})$

$= O(s2^{O(s)})$

$= O(2^{O(s)+s})$

$= O(2^{O(s)})$ No decider can repeat an ID, so time $\leq$ # of IDs

$\therefore$ PSPACE $= \bigcup_k \mathrm{SPACE}(n^k)$

$\subseteq \bigcup_k \mathrm{DTIME}(O(2^{O(n^k)}))$

$= \mathrm{DTIME}(2^{\mathrm{poly}(n)}) =$ EXP

So:

P $\subseteq$ PSPACE $\subseteq$ EXP.

<u>Prop</u>: NP $\subseteq$ PSPACE.

<u>Proof</u>: Given a ptime verifier $V(x,y)$ running in ptime on input x, where $|y| = \mathrm{Poly}(|x|)$.
A PSPACE TM M can simulate $V(x,y)$ for all possible y, accepting if it ever finds that $V(x,y)$ accepts.
M can re-use the same space (polynomial in $|x|$) for each simulation.

<u>Open</u>: NP $\stackrel{?}{=}$ PSPACE

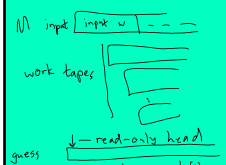
<u>Open</u>: PSPACE $\stackrel{?}{=}$ $\underbrace{\text{EXP}}_{\text{Time-bd}}$

<u>Known</u>: P $\neq$ EXP (proper)
(Time-hierarchy theorem)

<u>Known</u>: PSPACE $\neq$ EXPSPACE
(Space-hierarchy theorem)

<u>Nondeterministic</u> <u>space</u>:

TM M that is allowed one-way read-only access to a special tape, the <u>guess</u> <u>tape</u> (or <u>help</u> <u>tape</u>).



cannot move left; (only move right or stay put on each time step).

Given input w,
M accepts w iff there exist some contents of the guess tape that leads to M entering its accepting state.

[Could have defined NP using this model, where the proof string y would be read off the guess tape.]

Def: NPSPACE is the set of all languages that are decided by TMs with guess tapes as above that use space Poly(|input|) (not including the guess tape, which is read-only) and never loop for any input or guess tape contents.

Thm: For any reasonable space bound $s(n)$,

$$NSPACE(s(n)) \subseteq SPACE(s(n)^2)$$

[Savitch's Theorem]

Cor: NPSPACE = PSPACE.

Prop: $ALL_{NFA} \in PSPACE$

Idea: Guess a rejecting path
read the guess tape

Easy: PSPACE is closed under union, intersection, and complement.