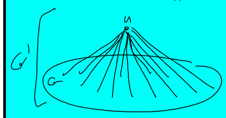
4 polynomial reductions
(3 easy, one medium)

Prop: 3-COL $\leq_p$ 4-COL

Proof: Given a graph G, we construct in ptime a graph G' such that G' is 4-colorable iff G is 3-colorable.



$G'.V := G.V \cup \{u\}$ where $u \notin G.V$

$G'.E := G.E \cup \{(u, x) : x \in G.V\}$

End construction.

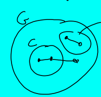
G 3-colorable $\Rightarrow$ G' is 4-colorable:
any 3-coloring of G can extend to a 4-coloring of G' by coloring u with a unused color

not the color of any vertex of G.

G' 4-colorable $\Rightarrow$ G is 3-colorable:
any 4-coloring of G' restricts to a 3-coloring of G, because all vertices of G are colored by some color unequal to the coloring of u. //

Note: Same construction p-reduces k-COL to $(k+1)$-COL for any k.

Def: Let G be a graph. A vertex cover of G is a set $C \subseteq G.V$ of vertices such that every edge of G has at least one endpoint in C.



can't happen if C is a v.c.

Def: VERTEX-COVER (VC) is the following decision problem:

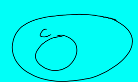
Instance: A pair $\langle G, K \rangle$ where G is a graph and K is an integer $0 \leq K \leq |G.V|$

Question: Does G have a vertex cover of size K?
$(\leq K?)$

Prop: IS $\leq_p$ VC.

Proof: Let G be a graph and let C be any set of vertices of G.
Notice that C is a v.c. iff $G.V - C$ is an indep set.

So, G has a v.c. of size K iff G has an independent set of size $|G.V| - K$.



Construction in ptime:
Given an instance $\langle G, K \rangle$ of IS, output $\langle G, |G.V| - K \rangle$.
Then G has an indep set of size K iff G has a v.c. of size $|G.V| - K$. //

Def: Let G be a graph.
A Hamiltonian path in G is a simple path (not a cycle) that passes through every vertex of G (exactly once, including the endpoints of the path.
A Hamiltonian circuit is a simple cycle in G that passes through each vx exactly once.

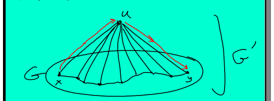

has a Ham path
no Ham circuit

Def: HP (resp. HC)
Instance: A graph G
Question: Does G have a Hamiltonian path (resp. Hamiltonian circuit)?

Know: HP & HC are both NP-complete.

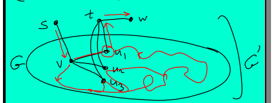
DHP & DHC are same but for digraphs.

Prop: $HP \leq_p HC$
Proof: Same reduction from 3-COL to 4-COL reduces HP to HC:



G has an H.P. $\Longleftrightarrow$ G' has an H.C. //

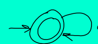
Prop: $HC \leq_p HP$.
Proof sketch: Given G, Construct G' as follows:



Def: $ALL_{NFA} :=$
$\{\langle N\rangle : N$ is an NFA that accepts all its input strings$\}$.

Prop: $3\text{-SAT} \leq_p \overline{ALL_{NFA}}$
Cor: NFA minimization is not ptime computable unless $P=NP$.
Proof of Cor:

$\langle N\rangle \in ALL_{NFA}$ iff its minimal equiv NFA is



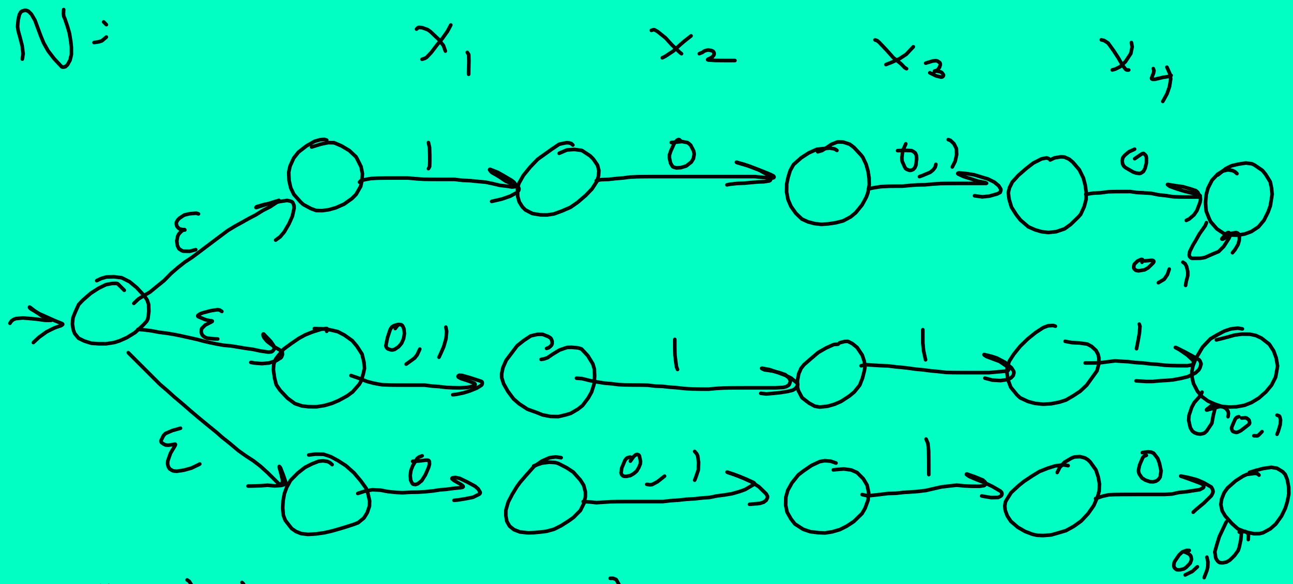
So if NFA min is easy, then so is ALL_{NFA}
($ALL_{NFA} \in P$)
$\therefore \overline{ALL_{NFA}} \in P$
$\therefore$ 3-SAT $\in P$
$\therefore P=NP$.

Proof of the Prop:
Given a 3cnf formula φ with variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_k$, we construct an NFA N with binary input alphabet, so that N accepts all binary strings except strings of length n encoding satisfying assignments to φ.

That is
$\overset{x_1 x_2}{10110}\ldots$
encodes $x_1 := 1$, $x_2 := 0$, $x_3 := 1$, ...

Ex: $\varphi := (\overline{x_1} \vee x_2 \vee x_4) \wedge$
$(\overline{x_2} \vee \overline{x_3} \vee \overline{x_4}) \wedge$
$(x_1 \vee \overline{x_3} \vee x_4)$

N:



All states are accepting.