Last time:

- CNF-SAT is NP-complete
- CNF-SAT $\leq_p$ 3-SAT

$\therefore$ 3-SAT is NP-complete

Today: More NP-complete problems

INDEPENDENT SET (IS):

Instance: A graph G and a number K s.t. $0 \leq K \leq |G.V|$

Question: Does G have an independent set of size K?

[An independent set of a graph G is a set of vertices, no two of which are adjacent (aka an "anticlique").

Prop: IS is NP-complete.

Proof: We show that 3-SAT $\leq_p$ IS.
$\therefore$ IS is NP-hard.
[IS $\in$ NP, "clearly"]

Let φ be a 3-cnf Boolean formula. Given input φ, we construct (in ptime) a graph G and number K s.t.

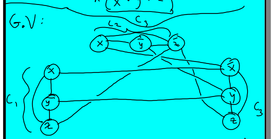
φ is satisfiable $\Longleftrightarrow$ G has an indep. set of size K.

Let $\varphi = C_1 \wedge \cdots \wedge C_n$ clauses, each having 3 literals.
We construct graph G as follows:

G has $3n$ vertices, 3 vertices for each clause, $\underbrace{\text{representing}}_{\text{and labeled with}}$ the 3 literals in the clause.

Ex: $\varphi = \overbrace{(x \vee \bar{y} \vee z)}^{C_1} \wedge \overbrace{(x \vee y \vee \bar{z})}^{C_2} \wedge \underbrace{(\bar{x} \vee y \vee z)}_{C_3}$

G.V:



Edges of G are of two kinds:

1. any two vertices in the same clause are adjacent.

[note: no IS of G can have more than one vertex in each clause.
$\therefore$ biggest possible i.s. has size n
set $K := n$.

2. Any two vertices labeled with contradictory literals are adjacent, (e.g. $x, \bar{x}$ or $y, \bar{y}$, etc.)

Output $\langle G, K = n \rangle$

Map from φ to $\langle G, K \rangle$ is clearly ptime computable (in FP)

Remains to show that

φ is sat $\Longleftrightarrow$ G has an i.s. of size $n = K$.

($\Longrightarrow$): Let $\vec{a}$ be a satisfying assignment for φ. Then $\vec{a}$ makes at least one literal true in each clause. Pick one true literal in each clause as the set S of vertices. Then $|S| = n$ and S is an indep set:

1. no two vertices chosen from the same clause
2. no vertices labeled with contradictory literals in S (because such literals cannot both be made true by the assignment.

(1) $\Rightarrow$ no type-1 edges in S
(2) $\Rightarrow$ no type-2 edges in S
$\therefore$ S is an indep set of size $K = n$.

($\Longleftarrow$) Assume G has an indep set T of size n. Because of type-1 edges, T has exactly one vertex in each clause. Use T to build a satisfying assignment for φ:

a) If some literal, say ℓ, is in T, set the assignment to make ℓ true. [can do this b/c of type-2 edges]
[e.g., if $\ell = x$, set $x := 1$;
if $\ell = \bar{x}$, set $x := 0$]

b) any variables left unset in (a) can be set to 1.

Let $\vec{b}$ be the corresponding truth assignment. Then $\vec{b}$ satisfies φ: All literals in T are set to 1, and T contains a literal in every clause, so each clause has ≥ 1 true literal. $\therefore$ φ is sat. ▨

Prop: CLIQUE is NP-complete.

Proof: CLIQUE $\in$ NP (clear).
To show NP-hardness, show
IS $\leq_p$ CLIQUE.

[Then Any NP set $A \leq_p$ CNF-SAT $\leq_p$ 3-SAT $\leq_p$ IS $\leq_p$ CLIQUE]

Given a graph G and number $0 \leq K \leq n$ ($n = |G.V|$), we output (in ptime) a pair G' and K' such that G has an indep set of size K iff G' has a clique of size K'.

Let G' be such that

$$G'.V = G.V$$

$$G'.E = \{(u,v) : u,v \in G.V \ \& \ u \neq v \ \& \ (u,v) \notin G.E\}$$

So 2 vertices are adjacent in G' iff they are not adjacent in G.

Thus, let $S \subseteq G.V = G'.V$.
S is an indep set in G iff S is a clique in G'.

Set $K' := K$.
Then G has an ind. set of size K iff G' has a clique of size K [the same set].
This construction is clearly polytime, so we are done. //

Def: A graph G is <u>k-colorable</u> ($k \geq 1$) if the vertices of G can be be assigned colors from a pallette of k colors so that no two adjacent vertices are assigned the same color.

Def: <u>3-COLORABILITY</u> (<u>3-COL</u>) is the problem

<u>Instance</u>: a graph G.
<u>Question</u>: Is G 3-colorable?

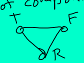
Prop: 3-COL is NP-complete.

Proof: 3-COL is in NP (clearly). For NP-hardness, we show that 3-SAT $\leq_p$ 3-COL

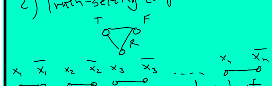
Given a 3-cnf fmla φ with variables $x_1, \dots, x_n$ and clauses $C_1 \wedge \cdots \wedge C_k = \varphi$, we construct in ptime a graph G s.t. G is 3-colorable iff φ is satisfiable.

G has 3 kinds of components

1) A triangle

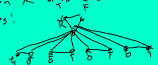


2) Truth-setting components:



with edges between R and each of thes 2n vertices:

e.g.



3) Sat-testing components:

For each clause $C_i = (\ell_1 \vee \ell_2 \vee \ell_3)$ have the following