Def:

$FIN_{TM} := \{\langle M\rangle : M \text{ is a TM } \& \ L(M) \text{ is finite}\}$

$INF_{TM} := \{\langle M\rangle : \text{- - - - - - - - - - infinite}\}$

Essentially, $INF_{TM} = \overline{FIN_{TM}}$ if every string encodes some TM.

Prop1: $A_{TM} \leq_m FIN_{TM}$

Prop2: $\overline{A_{TM}} \leq_m FIN_{TM}$

(equivalently, $A_{TM} \leq_m INF_{TM}$)

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Proof of Prop 2:

Let

$f :=$ "On input $\langle M, w\rangle$ where $M$ is a TM and $w$ is a string:

1. Let
   $R :=$ 'On input $x$:
   a) Run $M$ on input $w$'
2. Output $\langle R\rangle$"

$f$ is computable, and for all TMs $M$ & strings $w$,

$\langle M, w\rangle \in A_{TM} \implies M$ accepts $w$

$\implies R$ accepts all its inputs

$\implies L(R)$ is infinite

$\implies \underbrace{\langle R\rangle}_{f(\langle M,w\rangle)} \in INF_{TM}$

Conversely,

$\langle M, w\rangle \notin A_{TM} \implies M$ does not accept $w$

$\implies R$ accepts no strings

$\implies L(R)$ is finite
$[L(R) = \emptyset]$

$\implies \underbrace{\langle R\rangle}_{f(\langle M,w\rangle)} \in INF_{TM}$

$\therefore f$ m-reduces $A_{TM}$ to $INF_{TM}$ //

Proof of Prop 1:

Let

$f :=$ "On input $\langle M, w\rangle$ .....:

1. Let
   $R :=$ 'On input $x$:
   a) Run $M$ on input $w$ for $|x|$ many steps.
   b) If $M$ accepts $w$ in $\leq |x|$ steps, then <u>reject</u>; else <u>accept</u>'
2. Output $\langle R\rangle$"

$f$ is computable.

Given $M$ & $w$

<u>Case 1</u>: $M$ accepts $w$ ($\langle M, w\rangle \in A_{TM}$). Let $t$ be the number of steps it takes $M$ to accept $w$. Then $R$ rejects all strings $x$ with $|x| > t$. Thus $L(R)$ is finite, so $\langle R\rangle \in FIN_{TM}$
$\quad \| $
$f(\langle M, w\rangle)$

Conversely, if $M$ does not accept $w$ ($\langle M, w\rangle \notin A_{TM}$), then $R$ accepts all inputs $x$. $\therefore L(R)$ is infinite. $\therefore \langle R\rangle \notin FIN_{TM}$. $\therefore f$ m-reduces $A_{TM}$ to $FIN_{TM}$ //

The editing problem

Definition: Fix an alphabet $\Sigma$. An editing system over $\Sigma$ is a pair

$E := \langle \Sigma, P \rangle$ where

$P$ is a finite set of pairs of strings over $\Sigma$.

So $P = \{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}$.

[$E$ is also called a universal grammar.]

Def: Given $E = \langle \Sigma, P \rangle$ editing system and $w \in \Sigma^*$, an edit (of $w$ with respect to $E$) is a string $z$ obtained from $w$ by replacing some substring $x$ in $w$ with a string $y$, such that $(x, y) \in P$.

[Say $w \Rightarrow z$.]

Def: The editing problem has input $\langle E, w \rangle$ where $E = \langle \Sigma, P \rangle$ is an editing system and $w \in \Sigma^*$, and asks: Is there a finite sequence of edits, starting with $w$ and ending with $\varepsilon$?

$w \Rightarrow \cdots \Rightarrow \varepsilon$.

We let $EP_\Sigma$ be the editing problem over $\Sigma$.

As a language

$EP_\Sigma := \{\langle E, w \rangle : E$ is an editing system over $\Sigma$ and $w$ is a string over $\Sigma$ and $w$ edits to $\varepsilon$ in finitely many steps$\}$

Ex: $\Sigma = \{0, 1\}$

$P = \{(01, 100), (011, 110), (10, \varepsilon)\}$

$w = 0011$

$\underline{0\,01}1 \Rightarrow \underline{011}0 \Rightarrow 1\underline{10}0 \Rightarrow 10 \Rightarrow \varepsilon$