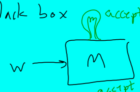
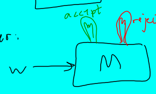
Recall:

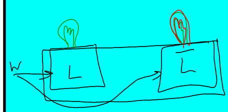
<u>Thm</u>: Let L be a language. If both L & $\overline{L}$ are T-rec, then L is decidable.

Think of a TM (recognizer) as a black box



Decider:





<u>Cor</u>: If L is T-rec and undecidable, then $\overline{L}$ is not T-rec.

<u>Cor</u>: $\overline{A_{TM}}$ is not T-rec

<u>Proof</u>: Shown A_{TM} is undecidable.

$A_{TM} = L(U)$ where recall that

U := "On input $\langle M, w\rangle$...:

1. Run M on input w [do what M does on w]

[If M accepts w, then U accepts $\langle M, w\rangle$ & conversely]

U — universal TM. //

Seen that A_{TM} & E_{TM} are undecidable.

$A_{TM} := \{\langle M, w\rangle :$ M is a TM that accepts w, i.e., $w \in L(M)\}$

$E_{TM} := \{\langle M\rangle : L(M) = \emptyset\}$

<u>Prop</u>: E_{TM} is not T-rec.

<u>Proof</u>: suffices to show that $\overline{E_{TM}}$ is T-rec.

$\overline{E_{TM}} = \{\langle M\rangle : L(M) \neq \emptyset\}$

$\cup \{w :$ w does not encode any TM$\}$

Recognizer for $\overline{E_{TM}}$:

N := "On input w:

1. If w does not encode a TM then <u>accept</u>.
2. Let TM M be such that $w = \langle M\rangle$
3. Cycling through all pairs $\langle x, t\rangle$ where x is a string over M's input alphabet and $t \geq 0$ (natural number),
   a) Run M on input x for t steps
   b) If M accepts x in $\leq t$ steps, then <u>accept</u>.

   [c) else, go on to the next pair] "

So $L(N) = \overline{E_{TM}}$

$\therefore \overline{E_{TM}}$ is T-rec.

$\therefore E_{TM}$ is not T-rec (by the theorem). ∎

<u>Def</u>:

$A_{\varepsilon,TM} := \{ \langle M \rangle : M \text{ is a TM that accepts } \varepsilon \}$

<u>Prop</u>: $A_{\varepsilon,TM}$ is T-rec.

<u>Proof</u>:

N = "On input $\langle M \rangle$, where M is a TM:

1. Run M on input ε [& do what M does]"

default, optional

Then M accepts ε iff N accepts $\langle M \rangle$

$\therefore L(N) = A_{\varepsilon,TM}$. //

<u>Prop</u>: $A_{\varepsilon,TM}$ is undecidable.

<u>Proof</u>: Suppose $A_{\varepsilon,TM}$ is decided by some TM D. Then use D to decide A_{TM} ↯:

M = "On input $\langle M, w \rangle$ where M is a TM & w is a string:

1. Let
   R := 'On input x:
   a) Run M on w'
2. Run D on input $\langle R \rangle$
3. If D accepts $\langle R \rangle$ then accept, else reject.

(ok because D is a decider)"

<u>Claim</u>: M decides A_{TM}. ↯

Given $\langle M, w \rangle$ input to A_{TM}:

$\langle M, w \rangle \in A_{TM} \iff M$ accepts w

$\Rightarrow R$ accepts all strings x

$\Rightarrow L(R) = \Sigma^*$

$\Rightarrow \varepsilon \in L(R)$

$\Rightarrow \langle R \rangle \in A_{\varepsilon,TM}$

$\Rightarrow D$ accepts $\langle R \rangle$

$\Rightarrow M$ accepts $\langle M, w \rangle$

$\langle M, w \rangle \notin A_{TM} \iff M$ does not accept w

$\Rightarrow R$ accepts none of its inputs

$\Rightarrow \varepsilon \notin L(R)$

$\Rightarrow \langle R \rangle \notin A_{\varepsilon,TM}$

$\Rightarrow D$ rejects $\langle R \rangle$

$\Rightarrow M$ rejects $\langle M, w \rangle$.

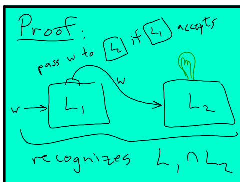
$\therefore M$ decides A_{TM} ↯

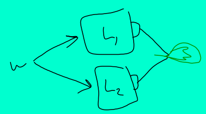
$\therefore A_{\varepsilon,TM}$ is undecidable. ∎

Properties of T-rec langs.

<u>Prop</u>: If L_1 & L_2 are T-rec, then so are $L_1 \cup L_2$ and $L_1 \cap L_2$.

Proof:





Run $\boxed{L_1}$ & $\boxed{L_2}$ in parallel, light lights if either $\boxed{L_1}$ or $\boxed{L_2}$ accepts. //

Thm: Every infinite T-rec. language includes an infinite decidable subset.

That is $\forall$ lang L,
L T-rec & infinite
$\Longrightarrow (\exists A)[A \subseteq L$, A is infinite, & A is decidable$]$.

Lemma: Let E be an enumerator with the following property: For any strings w, x, if E prints w then later prints x, then $|w| \leq |x|$.
Then $L(E)$ is decidable.

Proof:

Case 1: $L(E)$ is finite.
Then $L(E)$ is decidable (because all finite langs are decidable).

Case 2: $L(E)$ is infinite.
Then E prints arbitrarily long strings. Let

$D :=$ "On input w:
1. Run E until either
   (a) E prints w. Then accept
   (b) E prints some string x such that $|x| > |w|$ before (a) occurs. Then reject."

D decides $L(E)$:
For any w, if E prints w, then D accepts. If E does not print w, then eventually E prints some x such that $|x| > |w|$, so D rejects, but this correct, because then E will never print w. //

Proof of the Theorem:

Let L be infinite & T-rec.
Let E be an enumerator for L. Let E' be the following enumerator:

$E' :=$ "On no input:
1. Run E, maintaining a list l of all strings printed by E so far.
2. Whenever E prints a string w, check if $|w| \geq$ lengths of all string on list l. If so, print w; else continue."

[explanation deferred]

Twin primes conjecture:

There are infinitely many pairs of primes that differ by 2.

Open question currently.

Let

$L := \{ w \in \Sigma^* : \text{the twin prime conjecture holds} \}$

Which is the best answer:

1. L is decidable
2. L is undecidable
3. It is currently open whether L is decidable.