CSCE 551 / MATH 562

Today: Methods of proof

- Mathematical induction
- Proof by contradiction
- Proof by cases
- Proof by construction
- Pigeonhole principle

Example: In a room with $n \geq 2$ people, there are 2 different people who shake hands with the same number of other people.

Abstractly (theorem): Given any graph G on $n \geq 2$ vertices, there exist 2 distinct vertices with the same degree.

Proof:

Case 1: There exists a vertex v of G adjacent to all $n-1$ other vertices.
Thus every $v \in G.V$ has positive degree. That is, $\forall v \in G.V$, $1 \leq \deg(v) \leq n-1$

Fix v_0 s.t. $\deg(v_0) = n-1$.

Case 1a: There exists $v \neq v_0$ such that $\deg(v) = n-1$.
Done!

Case 1b: Not case 1a,
Then for every $v \neq v_0$, $1 \leq \deg(v) \leq n-2$.

deg maps $n-1$ vertices into the set $\{1, \dots, n-2\}$
$\therefore$ By the Pigeonhole principle, there exist $u \neq v_0$, $v \neq v_0$, $u \neq v$, $\deg(u) = \deg(v)$. Done.

$\therefore$ Case 1 implies the conclusion.

Case 2: Not case 1. That is, there is no $v \in G.V$ with $\deg(v) = n-1$.
Thus $\forall v \in G.V$,
$$\deg(v) \in \underbrace{\{0, \dots, n-2\}}_{\text{size } n-1}.$$

There are n vertices in $G.V$, so in this case, deg maps a set of size n into a set of size $n-1 < n$.
$\therefore$ Pigeonhole principle again, $\exists u \neq v$, $\deg(u) = \deg(v)$. Done

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Def'n: K_n = complete graph on n vertices (all vertices adjacent).

Prop: Let $c : K_6.E \to \{\text{red}, \text{blue}\}$ be any mapping. There exist distinct vertices u, v, w such that
$c(u,v) = c(u,w) = c(v,w)$.
(all red or all blue).

Think about proving this.
[Hint: Pigeonhole principle + proof by cases.]

A Generalization:
For every $k \geq 2$, there exists n such that, any 2-coloring of the edges of K_n contains a $\underbrace{\text{monochromatic}}_{\text{all edges the same color.}}$ complete subgraph of size k (K_k)

Def: Given k, $r(k,k)$ = the least such n.
(k'th Ramsey number)

Prop: $\forall n \geq 2$, any 2-coloring of K_n has a monochromatic K_k for some $k \geq \frac{1}{2} \log_2 n$

Theorem (Conway): Any set of 6 points in $\mathbb{R}^3$ in general position (no 4 points lie in the same plane), form two linked triangles.
corners of

Ex: Prove this.

Hint: K_5 is not planar (can't draw K_5 in the plane without two edges crossing.)

Course info:
3 sections:
I: Automata & Regular Languages
II: Turing machines, (un)decidability, T-recognizability
III: Resource-bounded computation, computational complexity.
difficulty.

Course homepage:
https://cse.sc.edu/~fenner/csce551/
links to the syllabus & some "announcements".

Six Quizzes, each worth 10% of your grade.
Other 40% is the final.
Homework ungraded, meant to prepare for the quizzes & exams.

Def: An alphabet is any nonempty finite set.
If Σ is an alphabet, we call the elements of Σ symbols or letters or characters.

Ex: $\Sigma = \{a, b, c\}$

$\Sigma = \{0, 1\}$ binary alphabet

If $|\Sigma| = 1$, this is a unary alphabet (e.g., $\Sigma = \{0\}$).

Def: Fix an alphabet Σ.
A string over Σ is a finite sequence of elements of Σ.

Ex: $\Sigma = \{a, b, c\}$
strings over Σ:

aa
abcb
caab
acab
ε — stands for the empty string, the unique string of length 0 (over any alphabet)

not a symbol from any alphabet.