Ex: $\Sigma = \{0,1\}$ assumed

$L := \{\langle R\rangle : R$ is a TM and $00 \in L(R)$ or $11 \notin L(R)\}$

$A_{TM} \leq_m L$

$\overline{A_{TM}} \leq_m L$

$A_{TM} \leq_m L$ via

$f :=$ "On input $\langle M, w\rangle$ ---- :

1. Let
   $R :=$ 'On input x :
   1. If $x = 11$ then accept.
   2. Run M on input w'
2. Output $\langle R\rangle$."

$\langle M, w\rangle \in A_{TM} \Rightarrow M$ accepts w
$\Rightarrow R$ accepts all strings x
$\Rightarrow L(R) = \Sigma^*$
$\Rightarrow 00 \in L(R)$
$\Rightarrow \underline{\langle R\rangle} \in L$
$\Rightarrow f(\underline{\langle M, w\rangle}) \in L$

$\langle M, w\rangle \notin L \Rightarrow M$ does not accept w
$\Rightarrow L(R) = \{11\}$
$\Rightarrow 11 \in L(R)$ and $00 \notin L(R)$
$\Rightarrow \langle R\rangle \notin L$
$\Rightarrow f(\langle M, w\rangle) \notin L$

$\therefore f$ m-reduces A_{TM} to L.

$\overline{A_{TM}} \leq_m L$ via g:

$g :=$ "On input $\langle M, w\rangle$ ----- :

1. Let
   $R :=$ 'On input x :
   1. If $x = 00$ reject.
   2. Else run M on input w'
2. Output $\langle R\rangle$."

$\langle M, w\rangle \in \overline{A_{TM}} \Rightarrow M$ does not accept w
$\Rightarrow R$ accepts no x
$\Rightarrow 11 \notin L(R)$
$\Rightarrow \langle R\rangle \in L$
$\Rightarrow g(\langle M, w\rangle) \in L$

$\langle M, w\rangle \notin \overline{A_{TM}} \Rightarrow M$ accepts w
$\Rightarrow L(R) = \Sigma^* \setminus \{00\}$
$\Rightarrow 11 \in L(R)$ and $00 \notin L(R)$
$\Rightarrow \langle R\rangle \notin L$
$\Rightarrow g(\langle M, w\rangle) \notin L$.

$\therefore g$ m-reduces $\overline{A_{TM}}$ to L. //

The Editing Problem (EP):

Def: Fix an alphabet Σ. An editing system (over Σ) is a finite set of pairs

$E = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$

where each $x_i, y_i \in \Sigma^*$

Given an editing system E as above and a string $w \in \Sigma^*$, an edit of w (via E) is a string w' obtained by replacing some substring of the form x_i in w with y_i. Say that $w \Rightarrow w'$ (w edits to w') if

$w = \alpha x_i \beta$ and $w' = \alpha y_i \beta$

for some $1 \leq i \leq n$ and strings $\alpha, \beta \in \Sigma^*$.

Def: The editing problem is the language

$EP := \{\langle \Sigma, E, w\rangle$: Σ is an alphabet, E is an editing system over Σ, and $w \in \Sigma^*$, and $\underbrace{w \Rightarrow \cdots \Rightarrow \varepsilon}_{\text{some finite sequence of edits}}\}$

Thm: EP is undecidable, in fact, $A_{TM} \leq_m EP$.

Proof idea: Given M and input w, Build an editing system ~~that~~ whose edits reflect the progression of IDs in a computation of M.