Recall: $L \subseteq \Sigma^*$.

DROP-ONE$(L) := \{ w : w$ is obtained from a string in L by removing a single symbol $\}$

Prop: L reg $\Rightarrow$ DROP-ONE(L) reg.

Proof sketch #2: Given a regex r over Σ, find a regex for DROP-ONE(r).

By induction on the syntax of r:

r	DROP-ONE(r)
$\emptyset$	$\emptyset$
$(a \in \Sigma)$ a	ε $(\varepsilon := \emptyset^*)$
(s, t regexes)	
$s \cup t$	$(\text{DROP-ONE}(s)) \cup (\text{DROP-ONE}(t))$
st	$(\text{DROP-ONE}(s))t \cup s(\text{DROP-ONE}(t))$
s^*	$s^*(\text{DROP-ONE}(s))s^*$

Correctness proof omitted.

Showing languages nonregular.

Fix Σ, $L \subseteq \Sigma^*$.

For $w \in \Sigma^*$, define the tail language

$$L_w := \{ x \in \Sigma^* : wx \in L \}$$

[Note: $L = L_\varepsilon$]

Theorem (Myhill-Nerode):

For a language $L \subseteq \Sigma^*$, define

$$\mathcal{C}_L := \{ L_w : w \in \Sigma^* \}.$$

Then L is regular if and only if $\mathcal{C}_L$ is finite. Moreover, if this is the case, then $\mathcal{C}_L$ forms the state set of the (unique) minimum DFA recognizing L.

Proof idea:

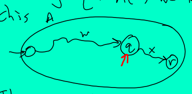
Suppose have a DFA A such that $L(A) = L$.

For any $w \in \Sigma^*$, recall that $A(w)$ is the state obtained by reading w.

Then (key idea)

$$L_w = L(A_{A(w)}).$$

Letting $q := A(w)$ we have this



Then $wx \in L$ iff r is accepting
iff $x \in L_w$ (by def of L_w)
iff $x \in L(A_q)$

So —

$$\mathcal{C}_L = \{ L_w : w \in \Sigma^* \}$$
$$= \{ L(A_q) : q \text{ state of } A \text{ reachable from the start state} \}$$

(assume same)

If A is not minimal, then $\exists$ pair of indist states q, r, i.e., $L(A_q) = L(A_r)$, so the mapping $q \mapsto L(A_q)$ (all states q) onto $\mathcal{C}_L$ is not 1-1, so $|\mathcal{C}_L| < |A\text{'s state set}|$

Suppose $\mathcal{C}_L$ is finite. Make a DFA $M = \langle \mathcal{C}_L, \Sigma, \delta, q_0, F \rangle$ where:

$$q_0 := L_\varepsilon \quad (= L)$$

$$F := \{ B \in \mathcal{C}_L : \varepsilon \in B \}$$

$\delta : \mathcal{C}_L \times \Sigma \to \mathcal{C}_L$ is defined as follows: Given $B \in \mathcal{C}_L$ and $a \in \Sigma$, Let w be some (any) string such that $B = L_w$. Define $\delta(B, a) = L_{wa}$.

δ is well defined:

$L_w = L_{w'} \Rightarrow L_{wa} = L_{w'a}$

$\forall x$

$x \in L_{wa} \Leftrightarrow wax \in L$

$\Leftrightarrow ax \in L_w$

$\Leftrightarrow ax \in L_{w'}$ (by assumption)

$\Leftrightarrow w'ax \in L$

$\Leftrightarrow x \in L_{w'a}$

$\therefore L_{wa} = L_{w'a}$. //

Left out: $L(M) = L$

also uniqueness of the min DFA.

Ex: $\Sigma = \{0,1\}$

$L := \{0^n 1^n : n \geq 0\}$

Prop: L is not regular.

Proof: L_ε contains ε

L_0 contains 1

L_{00} " 11

L_{000} " 111

$\vdots$

L_{0^n} contains 1^n but not 1^k for any $k \neq n$

$\therefore L_{0^m} \neq L_{0^n}$ if $m \neq n$

$\therefore \{L_{0^n} : n \geq 0\}$ is infinite $\subseteq \mathcal{C}_L$

$\therefore \mathcal{C}_L$ is infinite $\therefore L$ not reg by M-N thm

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Pumping Lemma (for regular languages)

Let L be any regular language. Then

$\exists p > 0$ (the pumping length)

$\forall s \in L$ with $|s| \geq p$

$\exists x, y, z$, ~~such that~~

1) $s = xyz$

2) $|xy| \leq p$

3) $|y| > 0$ (i.e., $y \neq \varepsilon$)

$\forall i \geq 0$,

$\underbrace{xy^i z}$ $\in L$.

$x\underbrace{yy\cdots y}_{i \text{ times}}z$ (pumping on y)

L is pumpable

L is not pumpable means

$\forall p > 0$

$\exists s \in L$, $|s| \geq p$,

$\forall x, y, z$ such that

$s = xyz$

$|xy| \leq p$

$|y| > 0$,

$\exists i \geq 0$, $xy^i z \notin L$.

Ex: $L := \{0^n 1^n : n \geq 0\}$

Show L is not pumpable:

Given $p > 0$,

let $s := 0^p 1^p$

[clearly $s \in L$ and $|s| = 2p \geq p$].

Given x, y, z such that $s = xyz$, $|xy| \leq p$, $|y| > 0$,

[so $y = 0^k$ for some $k > 0$].

Let $i := 0$.

[Then $xy^i z = xz$

$= 0^{p-k} 1^p \notin L$ ($p-k \neq p$)]

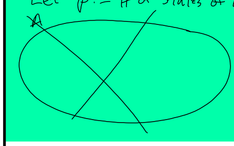
$\therefore L$ not pumpable

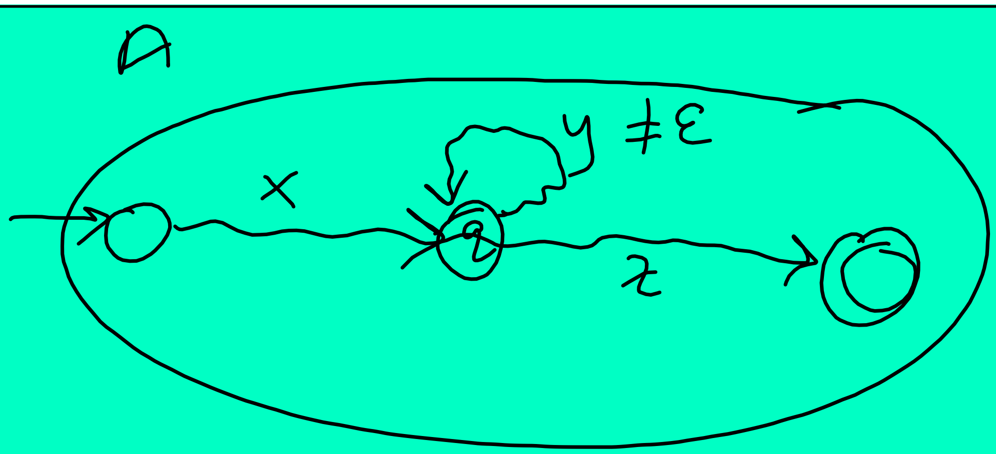
$\therefore L$ not regular (by Pumping Lemma)

Proof Sketch of pumping lemma.

Let L be regular, recognized by some DFA A.

Let $p :=$ # of states of A.





Let any $s \in L$ with $|s| \geq p$

Some state q is repeated among first p many transitions.

So: $y \neq \varepsilon$ and $|xy| \leq p$
and $s = xyz$

So $xy^iz \in L$ for all $i \geq 0$.

{go around the loop i times in a row}.