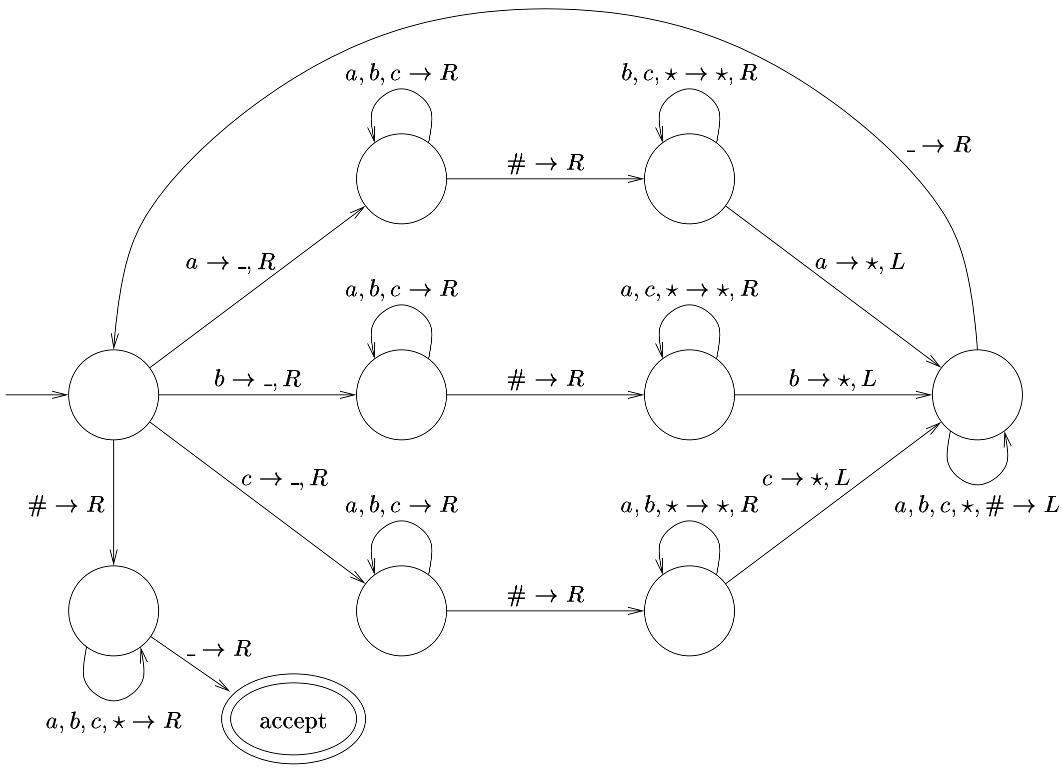
CSCE 551
Spring 2005
Final Exam

Do all problems, putting your answers in the exam book. There are 115 points total in the exam. Full credit for graduate students is 85 points; full credit for undergrads is 70 points. The rest is extra credit. You have 180 minutes.

For ease of reference, here are the definitions of several languages:

- $A_{\text{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts input } w\}$
- $E_{\text{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$
- $FIN_{\text{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) \text{ is finite}\}$

1. (2 points each; 16 points total) Consider the following standard 1-tape Turing machine M, with input alphabet $\{a, b, c, \#\}$ and tape alphabet $\{a, b, c, \#, \star, \_\}$, where we use the underscore ($\_$) to denote the blank symbol:



As usual, the rejecting state is omitted along with all edges leading to it. M is a decider, and the final loop just before the accept state ensures that M only accepts strings of the form $x\#y$ where $x, y \in \{a, b, c\}^*$.

(a) Which of the following strings are accepted by M? (The bullet is not part of the string.)

- $\#$
- $a\#$
- $b\#b$
- $ab\#bac$
- $ab\#cacbac$
- $aaba\#abacaba$
- $abc\#bcbcacacabab$
- $abc\#bbbbaccccbaaaacbbbb$

(b) (5 points) Characterize in brief, simple, and general terms the language decided by M. This does not mean describing how M behaves, only the net result.

2. (12 points) Give the state diagram for a standard 1-tape TM M with input alphabet $\{1, \#\}$ that, on input $1^i\#1^j$ where $i, j \geq 0$, accepts leaving just 1^{i+j} on the leftmost part of its tape. If the input is not of the above form, then M should halt and reject, but in this case we don't care what the final tape contents are. In either case, we don't care where M's tape head is when it halts. M should have no more than six (6) states, including both halting states. Do not show the rejecting state or any edges leading to it in your diagram. Thus, your diagram will have at most five vertices. (M computes the addition function over a unary alphabet.)

   You may either draw a transition graph or give transition table, listing the start state and halting states.

3. (2 points each; 12 points total) Consider the computable function f which, when given $\langle M, w\rangle$ as input where M is a TM and w a string, outputs the description $\langle N\rangle$ of a TM N which behaves as follows:

   $N =$ "On input x:
   1. Run M on input w for $|x|$ steps or until it halts, whichever comes first;
   2. If M accepts w within $|x|$ steps, then accept;
   3. Otherwise, reject."

   This function f is a mapping reduction of A_{TM} to which of the following languages? For each language, say "yes" or "no." NOTE: for this problem, we will assume that *every* string that f takes as input is the encoding $\langle M, w\rangle$ of *some* TM M and input string w.

   (a) E_{TM}
   (b) $\overline{E_{\text{TM}}}$
   (c) $\emptyset$
   (d) FIN_{TM}
   (e) $\overline{FIN_{\text{TM}}}$
   (f) $\{\langle N\rangle \mid N \text{ is a decider}\}$

   [Hint: What can you say about $L(N)$ if M accepts w? If M does not accept w?]

4. (15 points) Give an informal, high-level description of an algorithm that, given input $\langle M, w\rangle$ where M is a TM and w is a string, decides whether or not M will ever scan a blank tape cell during its computation with input w. That is, your algorithm decides the language

$$\{\langle M, w\rangle \mid \text{TM } M \text{ scans a blank cell sometime while computing on input } w\}.$$

[Hint: find a number m such that, if M ever does scan a blank cell, it must do so within m steps of the start of its computation. The number m will depend on the size of M's tape alphabet, the number of states of M, and the length of w.]

5. (15 points) Use the Pumping Lemma for regular languages to show that the unary language

$$\{0^{n(n-1)/2} \mid n > 0\}$$

is not regular. [Hint: The difference in length between $0^{n(n-1)/2}$ and the next string in the language is exactly n.]

6. (10 points) Let

$$REJ_{\mathrm{NFA}} = \{\langle N\rangle \mid N \text{ is an NFA that rejects at least one string}\}.$$

Describe the flaw in the following "proof" that $REJ_{\mathrm{NFA}} \in \mathbf{NP}$:

> Suppose we are given an NFA N as input. If N rejects at least one string, then the Prover can present such a string w to the Verifier, who can simulate N on input w in polynomial time (using the set-of-states technique described in class), and will thus accept, convinced that N rejects w and so $\langle N\rangle \in REJ_{\mathrm{NFA}}$. Conversely, if there is no such string w, then obviously the Prover cannot convince the Verifier that any such w exists, so the Verifier will always reject. Thus $REJ_{\mathrm{NFA}} \in \mathbf{NP}$.

(Actually, it is known that $REJ_{\mathrm{NFA}} \in \mathbf{PSPACE}$, but it is unknown whether $REJ_{\mathrm{NFA}} \in \mathbf{NP}$ or whether $REJ_{\mathrm{NFA}} \in \mathrm{co}\mathbf{NP}$.)

7. (15 points total) Let φ be the quantified Boolean formula

$$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 [(\overline{x_1} \vee x_2 \vee x_4) \wedge (x_3 \vee \overline{x_4} \vee \overline{x_5}) \wedge (x_1 \vee \overline{x_2} \vee x_5) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})].$$

   (a) (10 points) Using the polynomial-time mapping reduction from TQBF to GG (Generalized Geography) given in the book and in class, give the instance of GG (i.e., the graph) corresponding to φ. You may either draw the graph, or else give a list of vertices and a list edges.

   (b) (5 points) Is φ true? Explain.

8. (15 points) Let L be a language. Define the language

$$\forall\exists L = \{x \mid (\forall y, |y| = |x|)(\exists z, |z| = |x|)\ xyz \in L\}.$$

Show that if $L \in \mathbf{PSPACE}$, then $\forall\exists L \in \mathbf{PSPACE}$.