CSCE 355
2/16/2026

Pumping Lemma for Regular Languages

Def. Let $L \subseteq \Sigma^*$ be any language. L is <u>pumpable</u> if

there exists a $p > 0$ such that,

for every $s \in L$ such that $|s| \geq p$,

there exist strings $x, y, z \in \Sigma^*$ such that

a) $xyz = s$,

b) $|xy| \leq p$, and

c) $y \neq \varepsilon$ (equiv. $|y| > 0$)

and

for every $i \geq 0$,

$xy^iz \in L$.

Informally:

"~~Ev~~ Every sufficiently long string in L can be pumped arbitrarily, and the result is still in L."

$y \mapsto y^i$

Lemma (Pumping Lemma for Reg Langs): If L is regular, then L is pumpable, equiv, every regular language is pumpable.

Equiv: If L is not pumpable, then L is not regular (contrapositive).

A language L is not pumpable iff

pure predicate logic

$\forall p > 0$, [p is called the "pumping length"]

$\rightarrow \exists s \in L$ with $|s| \geq p$,

$\forall x, y, z \in \Sigma^*$ with $xyz = s$, $|xy| \leq p$, and $y \neq \varepsilon$,

$\exists i \geq 0$

$xy^iz \notin L$.

[$i := 1$ never works because $xy^1z = xyz = s \in L$]

[$i := 0$: "pumping down" $xy^0z = xz$]

[$i > 1$: "pumping up" $xy^3z = xyyyz$ e.g.]

Applications:

$L := \{ a^n b^n : n \geq 0 \}$

Prop: L is not pumpable.

Proof: Given $p > 0$, let $s := a^p b^p$. [$s \in L$ & $|s| = 2p \geq p$]

Given $x, y, z \in \{a, b\}^*$ such that $s = xyz$, $|xy| \leq p$, $y \neq \varepsilon$,

let $i := 0$. (pump down)

Claim that xy^0z $(= xz) \notin L$.

$y = a^k$ for some k,

and $k > 0$ because $y \neq \varepsilon$

Then $i := 0$ means k many a's are removed:

$$xy^0z = xz = a^{p-k}b^p \notin L \text{ because } p-k \neq p.$$

Thus L is not pumpable. //

$\therefore$ L is not regular (by the P.L.) //

[In fact any $i \neq 1$ in the above works.]

<u>Ex</u>: $L := \{a^m b^n : 0 \le m \le n\}$

<u>Prop</u>: L is not pumpable.

<u>Pf</u>. Given $p > 0$, let $s := \underline{a^p b^p}$ $[s \in L \;\&\; |s| \ge p]$

Given x, y, z s.t. $s = xyz$, $|xy| \le p$, $y \ne \varepsilon$,

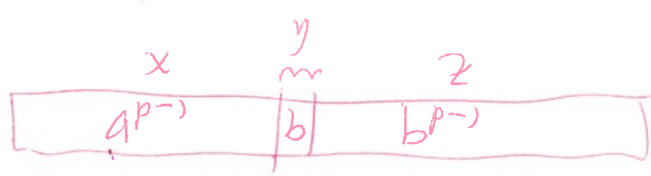
Let $i := \underline{2}$ (pump up)

<u>Explanation</u>: By choice of s, $y = a^k$ for some $k > 0$ $(y \neq \varepsilon)$

Then $xy^2z = xy\underline{y}z = a^{p+k}b^p \notin L$ because $p+k > p$. //

[choosing $s := a^{p-1}b^p \in L$ doesn't work.

Opponent: $x := a^{p-1}$
$y := b$
$z := b^{p-1}$



<u>No</u> value of i pumps out of the language.

<u>Ex</u>: $L = \{a^m b^n : 0 \le n \le m\}$

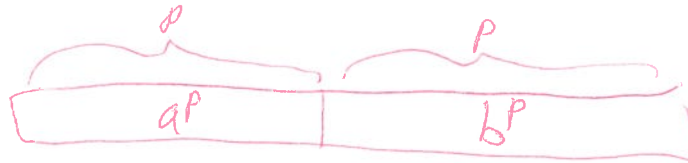
<u>Prop</u>: L is not pumpable.

Pf: Given $p > 0$, let $s := a^p b^p$

Given x, y, z s.t. $s = xyz$, $|xy| \le p$, $y \ne \varepsilon$,

Let $i := 0$

(only value that works)



As before $y = a^k$ (some $k > 0$).

Then $xy^0z = xz = a^{p-k}b^p \notin L$ because $p-k < p$.

[Choosing $s := a^p b^{p-1}$ is legal, but doesn't work: opponent chooses

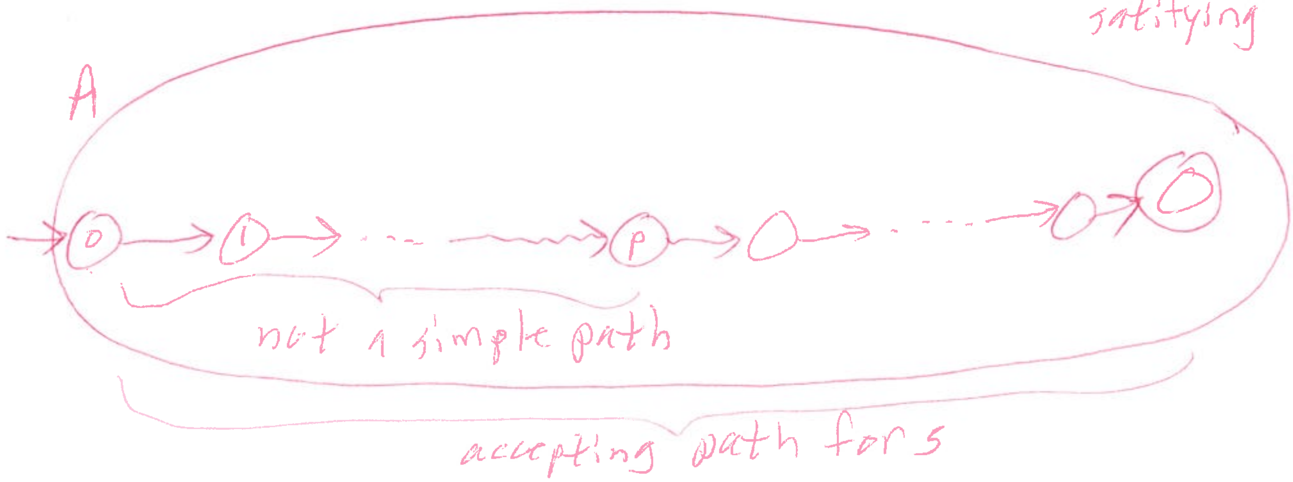
$x := \varepsilon$
$y := a$
$z := a^{p-1}b^{p-1}$

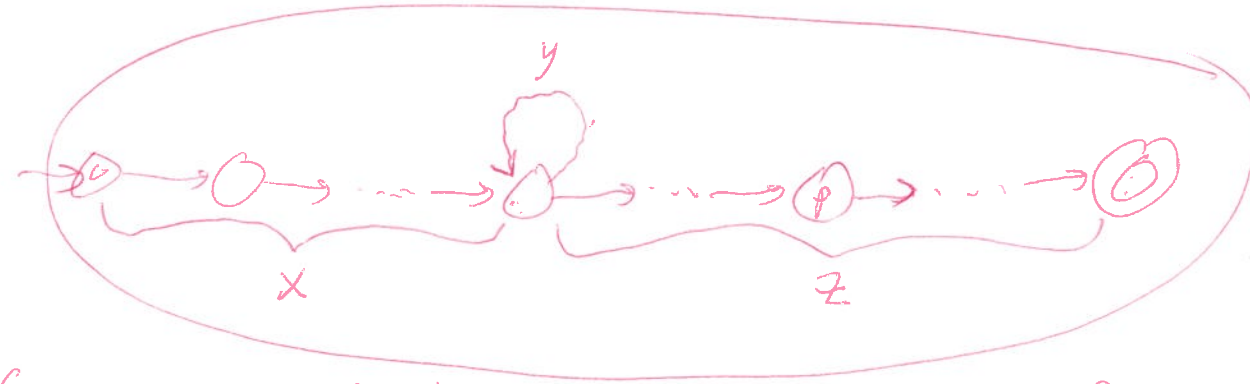
Then $xy^iz \in L$ for every $i \ge 0$, so opponent wins.]

Proof of the Pumping Lemma: Assume L is regular, let A be a DFA recognizing L.

Let p be the number of states of A.

Given any $s \in L$ with $|s| \ge p$, [need to choose x, y, z satisfying $s = xyz$, $|xy| \le p$, $y \ne \varepsilon$]





Can go around the loop any number of times (incl. 0 times): reading xy^iz get to the same accept state as s, while traversing the loop i times.

Know: $s = xyz$

$|xy| \le p$ because loop is among the 1st p many transitions

$y \neq \varepsilon$ b/c loop has ≥ 1 transition.

$\therefore$ $xy^iz \in L$ (accepted by A) for all $i \ge 0$. $\blacksquare$