CSCE 355
2/11/2026

More closure properties:

DROP-ONE, String Homomorphisms

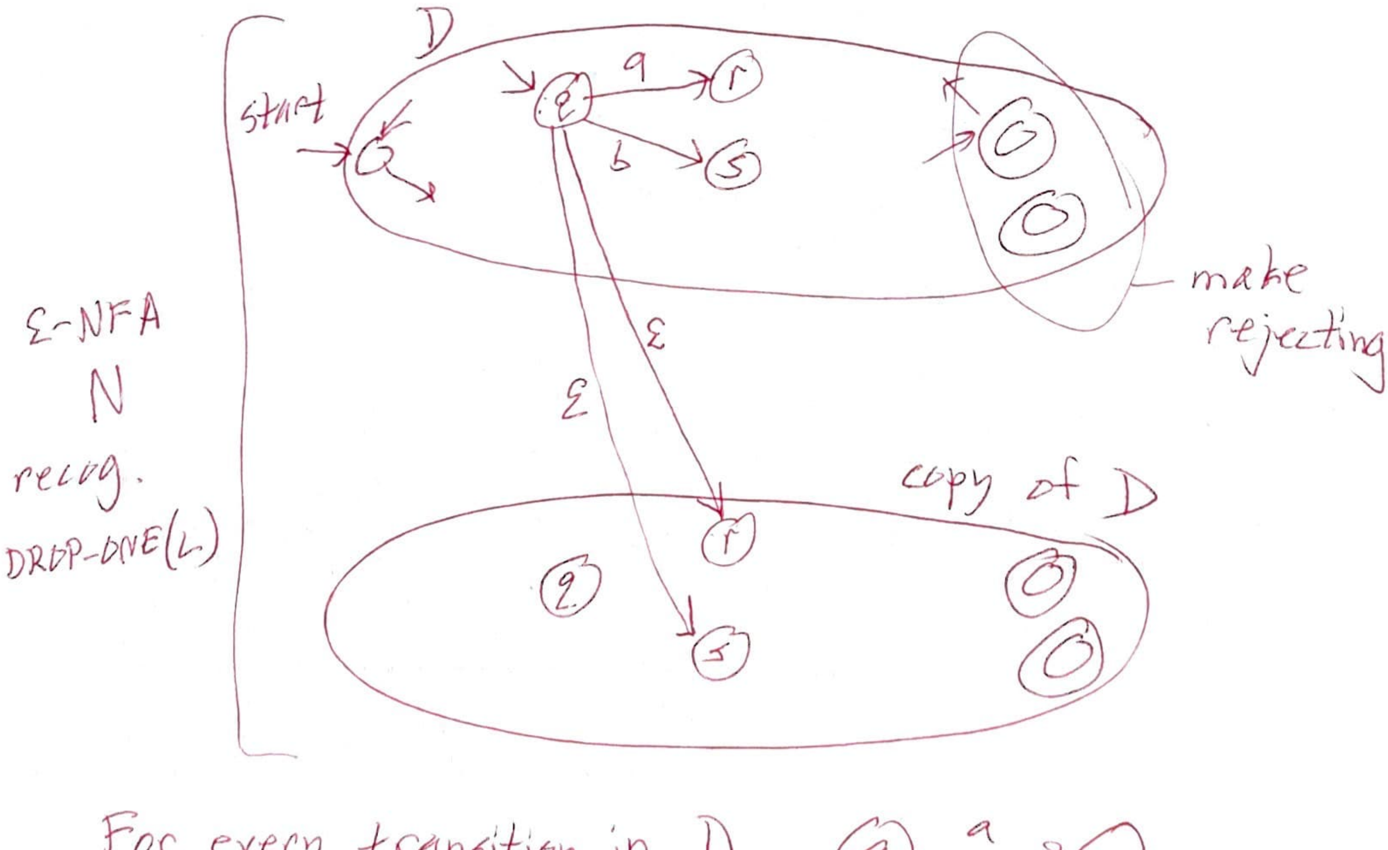
Given a language $L \subseteq \Sigma^*$, define

$$\text{DROP-ONE}(L) = \{wx : wax \in L \text{ for some } a \in \Sigma\}$$

$= \{$strings obtained from (nonempty) strings in L by removing a single char$\}$

__Prop__: If L is regular, then DROP-ONE(L) is regular.

__Proof 1__: Given a DFA D recognizing L, construct an ε-NFA recog. DROP-ONE(L) as follows:



For __every__ transition in D, $q \xrightarrow{a} r$, say add a ε-trans $\text{top } q \xrightarrow{\varepsilon} \text{bottom } r$

(Proof of correctness by hand-waving) //

Proof 2: Give rules for constructing a regex r' for DROP-ONE(L) given a regex r for L:

	r	r'
	$\emptyset$	$\emptyset$
$(a \in \Sigma)$	a	$\varepsilon \quad (= \emptyset^*)$
s, t regexes	$s + t$	$s' + t'$
	st	$st' + s't$
	s^*	$s^* s' s^*$

Correctness by handwaving. //

Def: For any language L, define

$$\sqrt{L} := \{ x : xx \in L \}$$

Prop: If L is regular, then $\sqrt{L}$ is regular. [Proof: exercise]

[By the way $\{ xx : x \in L \}$ may not be regular, even if L is reg.]

String homomorphisms:

Def: Let Σ and Γ be alphabets. A string homomorphism from Σ to Γ is a function

$$\varphi : \Sigma^* \longrightarrow \Gamma^*$$

that preserves concat, i.e., $\forall w, x \in \Sigma^*$,

$$\varphi(wx) = \underbrace{\varphi(w)\varphi(x)}_{\text{concat in } \Gamma^*}$$

Note: If φ is a str. homom., then $\varphi(\varepsilon) = \varepsilon$

Why? $\underbrace{\varphi(\varepsilon)}_{\text{length } n} = \varphi(\varepsilon\varepsilon) = \underbrace{\varphi(\varepsilon)\varphi(\varepsilon)}_{\text{length } 2n}$

$\therefore n = 2n \quad \therefore n = 0 \quad \therefore \varphi(\varepsilon) = \varepsilon$

Note: φ is completely determined by ~~what~~ how it maps strings of length 1. $\forall w = w_1 \cdots w_k$

$$\varphi(w) = \varphi(w_1 \cdots w_k) = \cdots = \varphi(w_1)\varphi(w_2)\cdots\varphi(w_k)$$

How φ maps symbols can be arbitrary.

Fix a string. homom. φ from Σ to Γ.

Given $L \subseteq \Sigma^*$, define (homomorphic image of L)

$$\varphi(L) := \{\varphi(w) : w \in L\}$$

Given $M \subseteq \Gamma^*$, define

$$\varphi^{-1}(M) := \{w \in \Sigma^* : \varphi(w) \in M\}$$

(inverse homom image of M)

Prop: Let Σ, Γ, φ be as above, $L \subseteq \Sigma^*$, $M \subseteq \Gamma^*$ as above.

1) If L is regular then $\varphi(L)$ is regular
2) If M " " " $\varphi^{-1}(M)$ is regular.

<u>Ex</u>: $\Sigma = \{a, b, c\}$. Given $L \subseteq \Sigma^*$, define
Add-b$(L) = \{w :$ w is obtained from a string in L by inserting a "b" after every "a" in the string$\}$

Add-b$\{caba, bb\} = \{ca\underline{b}ba\underline{b}, bb\}$

Add-b$(L) = \varphi(L)$ where φ is the str. homom. given by

$$\varphi(a) = ab$$
$$\varphi(b) = b$$
$$\varphi(c) = c$$

By the prop, add-b(L) is reg if L is regular.

<u>Proof</u> of the Prop:

1) Assume L is regular. We give rules for constructing a regex r' for $\varphi(L)$, given a regex r for L:

	r	r'
	$\emptyset$	$\emptyset$
$(a \in \Sigma)$	a	$\underbrace{\varphi(a)}_{\text{regex over } \Gamma}$
	$s+t$	$s'+t'$
	st	$s't'$
	s^*	$(s')^*$

$\varphi(\{a\}) = \{\varphi(a)\}$

Ex: $\varphi(L((ab^*c+dc)^*))$

$= L([(ab^*c+dc)^*]')$

$\varphi(a) = ca$
$\varphi(b) = bb$
$\varphi(c) = abc$
$\varphi(d) = \varepsilon$

$[(\underline{a}b^*c+dc)^*]' = (ca(bb)^*abc + abc)^*$

For any regex, substitute each symbol in the regex with φ of that symbol, preserving all the other ops // (1)

(2) Assume $M \subseteq \Gamma^*$ is regular. Let

$D = \langle Q, \Gamma, \delta, q_0, F \rangle$ be a DFA recognizing M.

We build a DFA $D' = \langle Q', \Sigma, \delta', q_0', F' \rangle$ recognizing $\varphi^{-1}(M)$ as follows:

$Q' := Q$

$q_0' := q_0$

$F' := F$

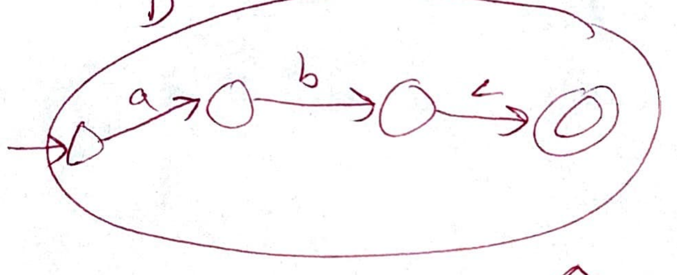
$\forall a \in \Sigma', \forall q \in Q$

define $\delta'(q, a) := \hat{\delta}(q, \varphi(a))$

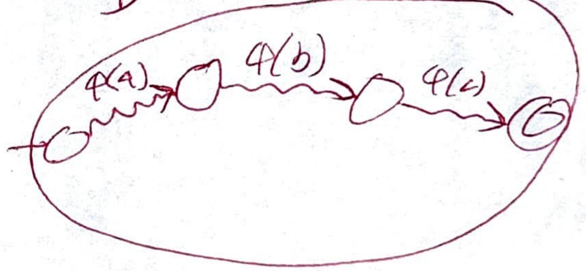
want D' to accept a string $w \in \Sigma'^*$ iff D accepts $\varphi(w)$

To read a symbol $a \in \Sigma'$, D' simulates D on $\varphi(a)$

D'



D



Can show by induction: $\underbrace{\hat{\delta'}(q, w)}_{\text{in } D'} = \underbrace{\hat{\delta}(q, \varphi(w))}_{\text{in } D}$

$\therefore$ D' accepts $w \iff D$ accepts $\varphi(w)$. //

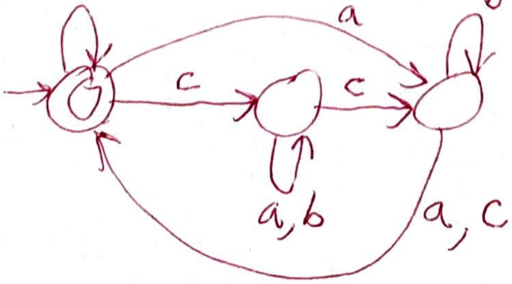
<u>Ex</u>: $\Sigma' = \{a, b, c\}$, $\Gamma = \{0, 1\}$, φ str. homom. given by

$\varphi(a) = 010$

$\varphi(b) = 11$

$\varphi(c) = 1010$

D'



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