CSCE 355
1/21/2026

①

Def: A language $L \subseteq \Sigma^*$ (Σ is an alphabet) is regular if some DFA recognizes L, i.e., $L = L(A)$ for some DFA A.

Last time: Complement construction: Given $A = \langle Q, \Sigma, \delta, q_0, F\rangle$ a DFA, define

$$\neg A := \langle Q, \Sigma, \delta, q_0, Q - F\rangle$$

$L(A) := \{x \in \Sigma^* : A \text{ accepts } x\}$

Then proved that $L(\neg A) = \overline{L(A)}$.

Thus the regular languages are closed under complement.

Next: Product construction: Given DFAs

$$A_1 := \langle Q_1, \Sigma, \delta_1, q_1, F_1\rangle$$

$$A_2 := \langle Q_2, \Sigma, \delta_2, q_2, F_2\rangle$$

let

$$A_1 \wedge A_2 := \langle Q, \Sigma, \delta, q_0, F\rangle, \text{ where}$$

$Q := Q_1 \times Q_2$ $\quad (:= \{(q, r) : q \in Q_1 \ \& \ r \in Q_2\})$

start state $q_0 := (q_1, q_2)$

$F := F_1 \times F_2$

and for every symbol $a \in \Sigma$ and states $q \in Q_1$ & $r \in Q_2$,

$$\delta((q, r), a) = (\delta_1(q, a), \delta_2(r, a)).$$

<u>Prop</u>: For any DFAs A_1 & A_2 with common input alphabet Σ,

$$L(A_1 \wedge A_2) = L(A_1) \cap L(A_2).$$

<u>Corrolary</u>: If L_1 & L_2 are regular, then so are:

$$\overline{L_2},\ \overline{L_1},\ L_1 \cap L_2,\ L_1 \cup L_2 = \overline{\overline{L_1} \cap \overline{L_2}} \quad \text{(De Morgan)}$$

$$L_1 - L_2 := L_1 \cap \overline{L_2}$$

$$L_1 \triangle L_2 := (L_1 - L_2) \cup (L_2 - L_1)$$

<u>Lemma</u>: Let A_1 & A_2 be as above, and let $x \in \Sigma^*$ be any string. Then, letting $A := A_1 \wedge A_2$ as above:

$$(*) \quad \underset{\substack{\uparrow \\ \text{in } A_1 \wedge A_2 \\ = A}}{\hat{\delta}}\big(\overbrace{(q_1, q_2)}^{\text{start state of } A}, x\big) = \big(\hat{\delta}_1(q_1, x), \hat{\delta}_2(q_2, x)\big).$$

<u>Proof</u>: By induction on x.

<u>Base case</u>: $x = \varepsilon$:

$$\hat{\delta}((q_1, q_2), \varepsilon) \underset{\text{def of } \hat{\delta}}{=} (q_1, q_2) \overset{\text{def of } \hat{\delta}_1 \,\&\, \hat{\delta}_2}{=} \big(\hat{\delta}_1(q_1, \varepsilon), \hat{\delta}(q_2, \varepsilon)\big) \checkmark$$

<u>Inductive case</u>: $x = ya$ where $y \in \Sigma^*$ is the principal prefix of x and $a \in \Sigma$ is the last symbol of x

③

[ind hypothesis: (*) holds for y] Then

$$\hat{\delta}((q_1,q_2),x) \overset{x=ya}{=} \hat{\delta}((q_1,q_2),ya)$$

$$\overset{\text{def of } \hat{\delta}}{=} \delta(\hat{\delta}((q_1,q_2),y),a)$$

$$\overset{\text{ind hyp.}}{=} \delta((\hat{\delta}_1(q_1,y),\hat{\delta}_2(q_2,y)),a)$$

$$\overset{\text{def of } \delta}{=} (\delta_1(\hat{\delta}_1(q_1,y),a),\ \delta_2(\hat{\delta}_2(q_2,y),a))$$

$$\overset{\text{defs of } \hat{\delta}_1 \,\&\, \hat{\delta}_2}{=} (\hat{\delta}_1(q_1,ya),\hat{\delta}_2(q_2,ya))$$

$$\overset{ya=x}{=} (\hat{\delta}_1(q_1,x),\hat{\delta}_2(q_2,x))$$

so (*) holds for x.

$\therefore$ By induction, (*) holds for any string x. $\boxed{//}$ Lemma

Prop (again): $L(A) = \cancel{L(A)}\ L(A_1) \cap L(A_2)$

Proof: Let x be any string. (over Σ)

$$x \in L(A) \iff \text{A accepts } x$$

$$\iff \hat{\delta}((q_1,q_2),x) \in F_1 \times F_2$$

$$\iff (\delta_1(q_1,x),\delta_2(q_2,x)) \in F_1 \times F_2$$

by the lemma: (*) holds

$\iff \delta_1(q_1, x) \in F_1$ and $\delta_2(q_2, x) \in F_2$

$\iff A_1$ accepts x and A_2 accepts x

$\iff x \in L(A_1) \cap L(A_2)$.

Since x was arbitrary, this implies $L(A) = L(A_1) \cap L(A_2)$ $\blacksquare$

<u>Ex</u>: $L := \{x \in \{0,1\}^* : x$ ends with 0 <u>and</u> has an odd number of 1's$\}$

$L_1 := \{x : x \text{ ends in } 0\}$

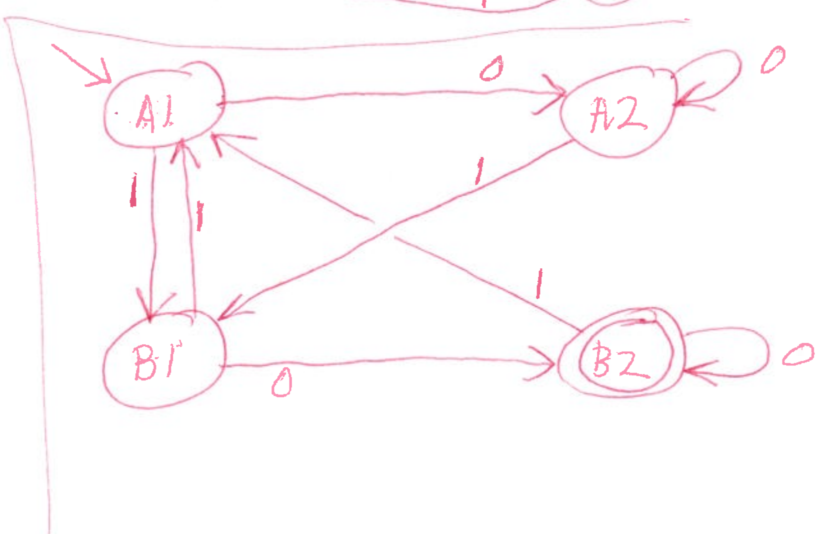
$L_2 := \{x : x \text{ has an odd \# of 1's}\}$

Then $L = L_1 \cap L_2$.

$L_1 = L(A_1)$ where

A_1
$=$

$L_2 = L(A_2)$ where $A_2 =$



E: $\Sigma = \{a, b\}$

$L = \{x \in \Sigma^* : x$ has ab as a substring but not ba as a substring$\}$

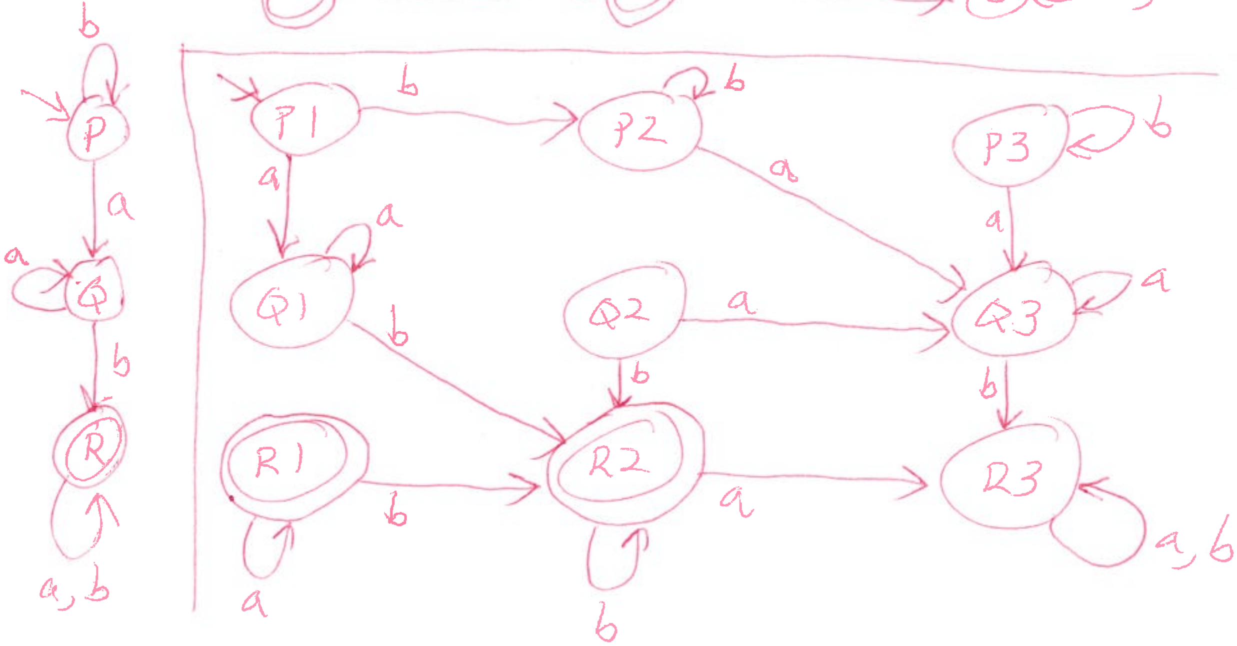
$L = L_1 \cap \overline{L_2}$ where

$L(A_1) = L_1 := \{x : x$ has ab as a substring$\}$

$L(A_2) = L_2 := \{x : \ldots ba \ldots\}$

where

$\neg A_2$

$A_1 =$ 

Can remove R1, P3, Q2 — these are unreachable from the start state. 6 states left.

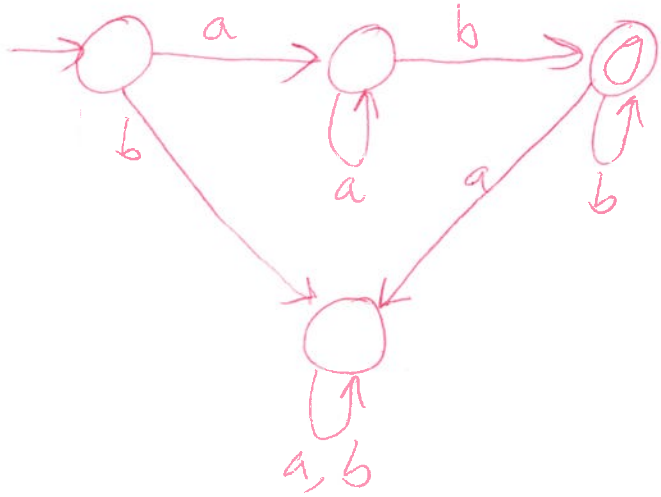
[Def: A DFA is sane if every state is reachable from the start state.]

DFA recognizing L.

$x \in L$ means $x = \underbrace{\cdots}_{\text{all } a\text{'s}} ab \underbrace{\cdots}_{\text{all } b\text{'s}}$

(but can't have ba)

$L = \{a^m b^n : m > 0 \text{ and } n > 0\}$



4 states