CSCE 355
4/21/2025

Theorem: The editing problem (EP) is undecidable.

Given an alphabet Σ (fixed)
Given a finite set $E = \{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}$
where $x_1, \dots, x_n, y_1, \dots, y_n \in \Sigma^*$.

A string $w \in \Sigma^*$ edits to a string $w' \in \Sigma^*$ if w' is obtained from w by replacing some substring x_i of w with y_i.
Write $w \Rightarrow w'$.

Def: Fix E. $w \Rightarrow w'$ (with resp. to E) if $\exists$ strings $u, v \in \Sigma^*$ and $1 \le i \le n$ such that

$$w = u x_i v \quad \text{and}$$
$$w' = u y_i v$$

Def. Editing problem: Given an editing system E and a string w, is there a finite sequence of edits starting with w and ending with ε?

$$w \Rightarrow \cdots \Rightarrow \varepsilon \,?$$

Ex: $\Sigma = \{0,1\}$. $E = \{(10, 01), (01, 1), (11111, \varepsilon)\}$

$w = 11111000$ yes

$w = 1111000$ no

Proof: Idea: Given a TM M and an input string w, we ~~produce~~ construct (algorithmically) an editing system E and string w' such that

$$\underbrace{M \text{ accepts } w}_{\text{decider}} \iff \underbrace{w' \text{ edits to } \varepsilon \text{ with resp. to } E}_{\text{decider}}$$

Construction: Let $M = \langle Q, \Sigma, \Gamma, \delta, q_0, B, F \rangle$

The alphabet Σ' for E & w' is as follows

$\Sigma' = \Gamma \,\dot\cup\, Q \,\dot\cup\, \{\$, \downarrow\}$

$w' = \$ q_0 w \$$

E consists of the following pairs:

(Right head movement)

1. For every $q \in Q$ and $a \in \Gamma$ such that $\delta(q, a) = (r, b, \rightarrow)$ for some $r \in Q$ and $b \in \Gamma$, add the pair (qa, br) to E.

(Left head movement)

2. For every $q \in Q$ and $a \in \Gamma$ such that $\delta(q, a) = (r, b, \leftarrow)$, add the following pairs to E: for every $c \in \Gamma$, (cqa, rcb) to E

3. (Expansion) For every $q \in Q$, add the pairs $(q\$, qB\$)$ and $(\$q, \$Bq)$ to E (can pad an ID with blanks on either side.)

4. ~~(Reductio~~ (Acceptance) For every $q \in F$ (accepting) and $a \in \Gamma$ such that $\delta(q, a)$ is undefined (halting), add the pair $(qa, \downarrow)$ to E.

5. (Reduction) For every $a \in \Gamma$, add the pairs

$(\downarrow a, \downarrow)$ and $(a\downarrow, \downarrow)$ to E

6. Add the pair $(\$\downarrow\$, \varepsilon)$ to E.

End-of-construction

By the discussion, M accepts w iff

$$w' \Rightarrow \cdots \Rightarrow \$\ldots\downarrow\ldots\$ \Rightarrow \cdots \Rightarrow \varepsilon$$

So deciding the editing problem allows us to decide A_{TM} ↯. Thus EP is undecidable. //

Note: The Length-nonincreasing editing problem restricts EP ~~so that th~~ to the instances where $|y_i| \leq |x_i|$ for all pairs $(x_i, y_i) \in E$. This restricted problem is decidable.

Def: The halting problem for TMs ~~HALT~~

$$HALT_{TM} = \{ (M, w) : M \text{ is a TM that halts on input } w \}$$

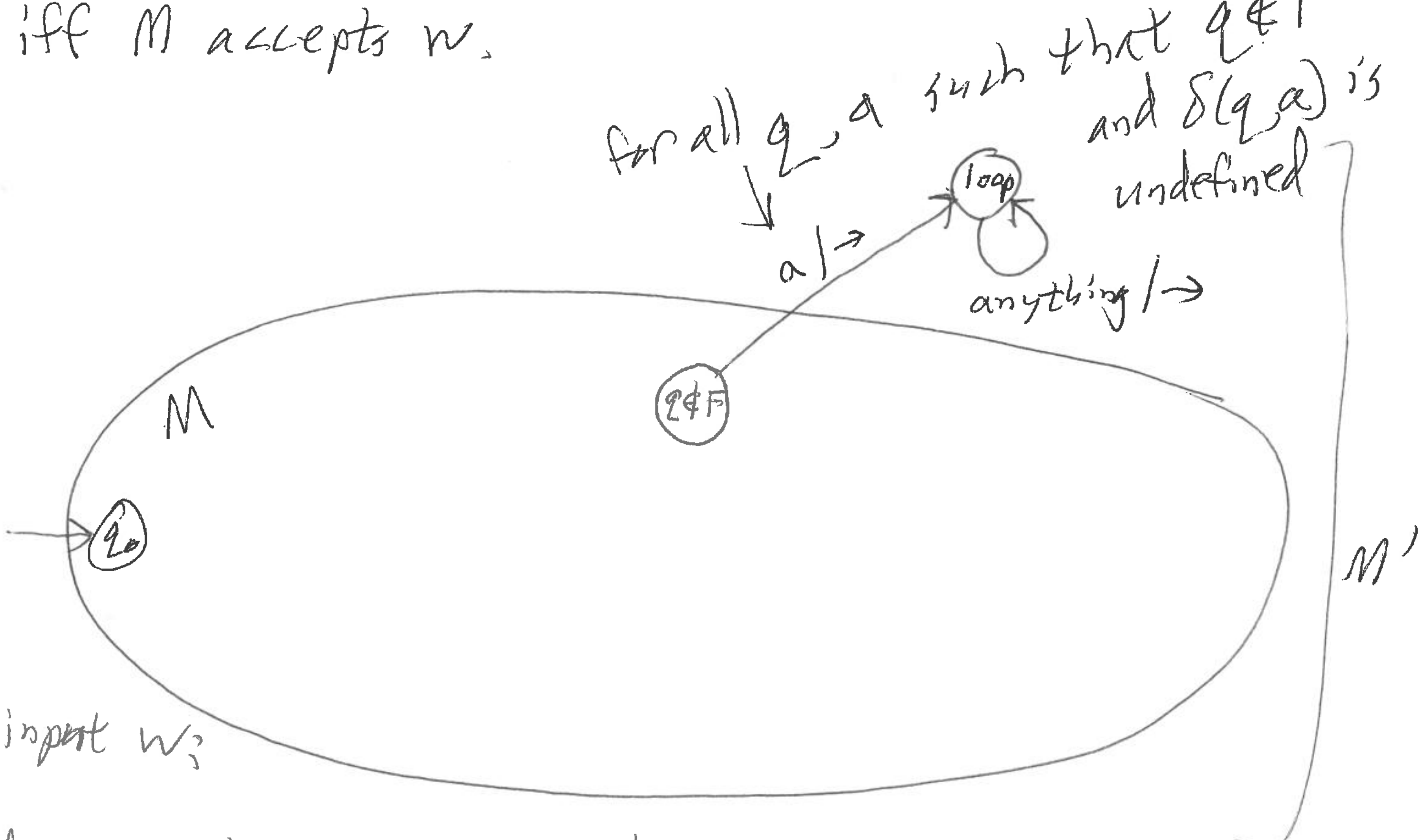
Prop: $HALT_{TM}$ is undecidable.

Proof: We can convert any decision procedure for $HALT_{TM}$ into a decision procedure for A_{TM}.

$\lightning$ = "contradiction"

($\Rightarrow\Leftarrow$ means the same)

Given TM M, ~~and input string w,~~ construct a TM M' such that M' halts on an input w iff M accepts w.



$\forall$ input w;

M accepts w $\Rightarrow$ M' accepts w (extra transitions are not taken)

M loops on w $\Rightarrow$ M' loops on w (" " " " ")

M rejects w $\Rightarrow$ M' loops on w

$\therefore$ $\underbrace{M \text{ accepts } w}_{\text{have decider}}$ $\Longleftrightarrow$ $\underbrace{M' \text{ halts on } w}_{\text{have decider}}$

have decider $\Leftarrow$ have decider

$\lightning$

Def: ALL_{CFG} is the following decision problem:

Given a CFG G, is it true that $L(G) = \Sigma^*$, where Σ is the ~~tk~~ terminal alphabet of G?

~~Theor~~ ~~G~~

$S \to 0S1 \mid \varepsilon$ no

$S \to 0S1 \mid 1S0 \mid \varepsilon$ no

$S \to 0S0 \mid 0S1 \mid 1S0 \mid 1S1 \mid \varepsilon$ no (only even-length strings)

$S \to 0S0 \mid 0S1 \mid 1S0 \mid 1S1 \mid 0 \mid 1 \mid \varepsilon$ yes

$S \to 0S \mid 1S \mid \varepsilon$ yes

Proof outline: Given any TM M and input w

1) we construct (algorithmically) a PDA P such that P ~~accepts~~ rejects just those strings that encode the accepting computation of M on w (in a certain way), if M accepts w.
If M does not accept w, then P accepts all strings.

2) Convert P into an equivalent grammar G (algorithmically — we already know how to do this)

M accepts $w \iff L(G) \neq \Sigma^*$

$\underbrace{M \text{ does not accept } w}_{\text{decision procedure}} \iff \underbrace{L(G) = \Sigma^*}_{\text{decision proc.}}$

decision procedure ⟵ decision proc.

Def: Given M and input w, a rigid accepting trace of M on input w is a string of the form

$\underline{ID_0 \vdash ID_1 \vdash \cdots \vdash ID_n}$ for some n, such that

$ID_0 = q_0 w$ (initial config of M on input w)

ID_n is an accepting config of M on w

and $ID_i \vdash ID_{i+1}$ (all $0 \leq i < n$). Additionally, Each ID_i covers the same segment of the tape, for all i. (So all IDs have the same length).