CSCE 355
4/16/2025

Church-Turing thesis (cont.)

Already: - memory movement
- copying
- addition & subtraction, increment & decrement
- testing for 0

Multiplication

Input: x#y   ($x, y \in \{0,1\}^*$ numbers in binary)

Output: xy   (numerically)

~~while x ≠ 0~~

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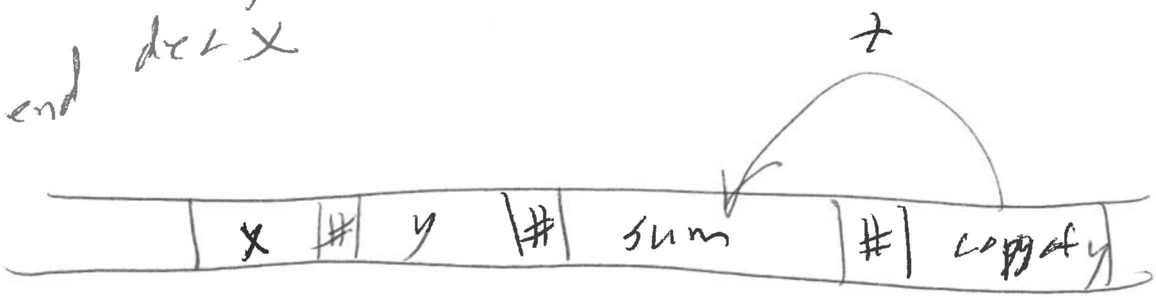
sum := 0
while x ≠ 0
    add y to sum
    dec x
end
```



Comparing x with y:

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while x ≠ 0 and y ≠ 0
    dec x
    dec y
end-while
```

If $x \neq 0$, goto $\longrightarrow$ $>$

else if $y \neq 0$ goto $\longrightarrow$ $<$

else goto $\longrightarrow$ $=$

Established the C.-T. thesis:

The TM model captures the notion of general-purpose computation.

Inputs (& outputs) of TMs are strings.

All types of data structures:

- lists, arrays
- trees, graphs, integers, floats, ...

can be effectively encoded as strings (binary strings, in fact).

Other finitely encodable object types:

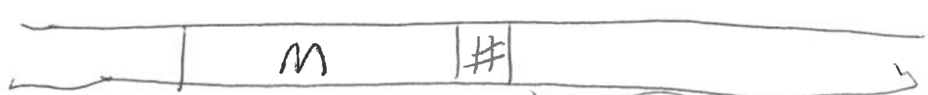
- DFAs
- NFAs
- ε-NFAs
- PDAs
- Grammars
- TMs

A TM could take another TM as input.

One possibility:

"On input M where M is a TM:

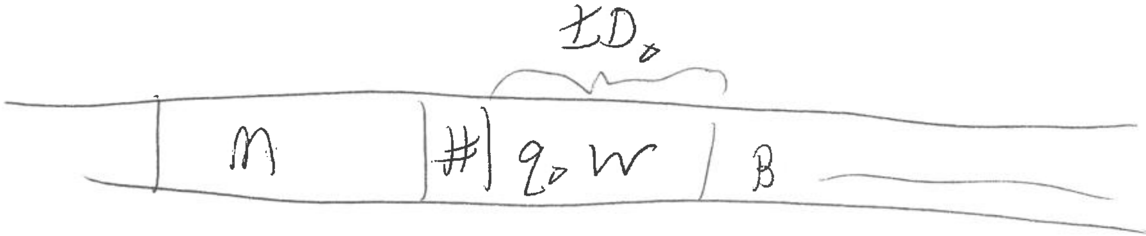
Simulate M on input ε"



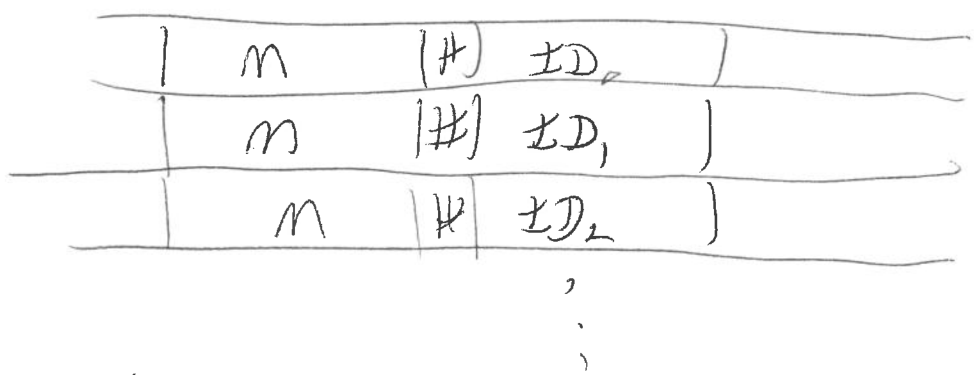
To simulate M, keep track of not just the tape contents of M but also the current state and head position

"On input M and w, where M is a TM and w is a string over M's input alphabet:

1. write the initial ID of M on input w on the tape;



2. ~~Whe~~ While the current ID has a successor, replace the current ID with the successor



where $ID_0 \vdash ID_1 \vdash ID_2 \vdash \ldots$

3. If current ID is accepting, then accept; else reject."

Shorthand:

U "On input (M, w) where M is a TM and w a string:

1. Run M on input w [& do what M does]"

M accepts w $\Longrightarrow$ U accepts (M, w)

M rejects w $\Longrightarrow$ U rejects (M, w)

M loops on w $\Longrightarrow$ U loops on (M, w)

U is called a universal TM.

U can simulate any TM on any input.

"On input (M, w, t) where M is a TM, w is a string, and t is a natural number:

1. Run M on w for t steps (or until M halts, whichever comes first).
2. If M ~~halt~~ halts in $\leq t$ steps, do what M does; else reject."

Def: A_{TM} is the language $\{(M,w) : M$ is a TM that accepts string $w\}$
(The acceptance problem (language) for TMs).

Theorem: A_{TM} is undecidable.

Proof: Consider the following algorithm (TM)
Suppose TM D that decides A_{TM}.

F := "On input M where M is a TM:

1. Form the pair (M,M) (F,F)
2. Run D on input $\boxed{(M,M)}$ (F,F)
3. If D accepts, then reject F
   If D rejects, then accept F,

Consider what F does on input F.

D accepts (F,F) means (by assumption) $(F,F) \in A_{TM}$
that is, F accepts input F,
but then F rejects F (by def of F)
contradition

D rejects (F,F) means $(F,F) \notin A_{TM}$, that is
F does not accept F.
But then F accepts F contradiction

Thus no such D can exist. $\blacksquare$

But A_{TM} is T-rec. (Recognized by the universal machine U.

$$\langle M,w\rangle \in A_{TM} \iff M \text{ accepts } w \iff U \text{ accepts } \langle M,w\rangle.$$

Theorem: Let L be any language. If L and $\overline{L}$ are both T-rec., then L is decidable.

Proof: Let M_1 & M_2 be TMs such that $L(M_1) = L$ and $L(M_2) = \overline{L}$. Let D be the following TM:

$D :=$ "On input w:

1. Run M_1 and M_2 "simultaneously" on w (step by step), until either M_1 or M_2 accepts w.
2. If M_1 accepts w, then accept; else reject"

D is a decider that accepts w iff M_1 accepts w, iff $w \in L$.

$\therefore L$ is decidable. //

Corollary: $\overline{A_{TM}}$ is not T-rec.

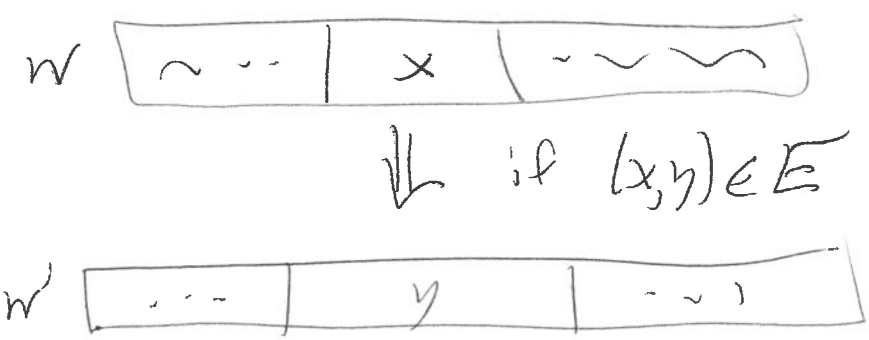
Proof: If $\overline{A_{TM}}$ is T-rec, then A_{TM} is decidable, by the last theorem, but A_{TM} is undecidable. //

The editing problem

Fix an alphabet Σ.

An editing system E over Σ is a finite set of pairs of strings $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ all over Σ.

Given an editing system E and string $w \in \Sigma^*$ an edit of w is a string $w' \in \Sigma^*$ resulting by replacing some substring x_i in w with y_i. So



We say $w \Rightarrow w'$

Editing ~~prog~~ Problem (EP)

Given an editing system E and an initial string w, is there a sequence of edits, starting with w and ending with ε?

$$w \Rightarrow \cdots\cdots \Rightarrow \varepsilon$$