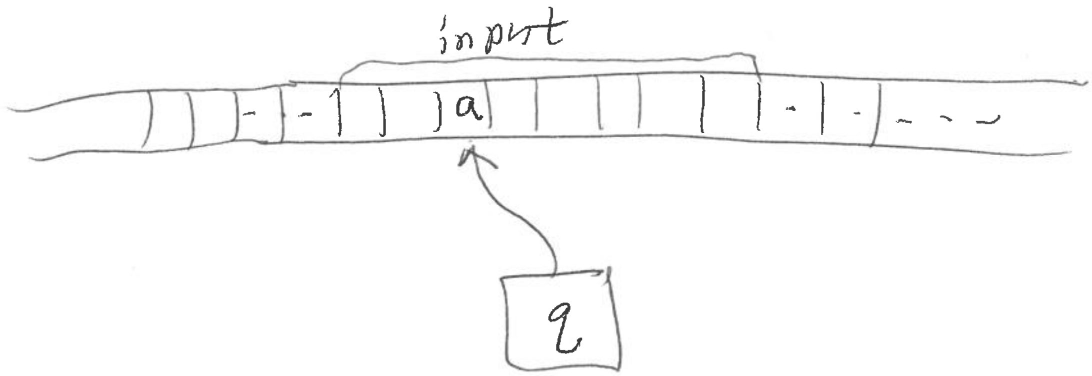
Turing Machines

~~Recognition vs. Decision (of languages)~~

A Turing machine is a finite-state model, like a DFA but ~~w~~ some added capabilities:

- scanning head can move ~~to~~ left or right
- can overwrite symbols being scanned
- input is on an infinite read/write "tape"



one symbol per cell

Def: A Turing machine (TM) is a tuple

$$\langle Q, \Sigma, \Gamma, \delta, q_0, B, F \rangle$$ where

- Q is a finite set (elements are states)
- Σ is an alphabet (the input alphabet; inputs to TM computations are strings over Σ)
- Γ is an alphabet (the tape alphabet; all cells of the tape at all times contain a symbol from Γ

such that $\Sigma \subseteq \Gamma$ and $\underbrace{Q \cap \Gamma = \emptyset}_{\text{convenient}}$ ②

- $q_0 \in Q$ (the start state)
- $B \in \Gamma \setminus \Sigma$ (the blank symbol, often written $\sqcup$ or just)
- $F \subseteq Q$ (the set of accepting states)
- δ is the transition function:

$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{\leftarrow, \rightarrow\}$$

partial — may be undefined on some arguments

end of Def

$$\delta(q, a) = (r, b, d)$$

$(q \in Q, a \in \Gamma)$

$r \in Q$

$b \in \Gamma$

$d \in \{\leftarrow, \rightarrow\}$

means: if the TM is in state q and scanning a, then in the next step:

- a is replaced by b
- state is r
- head move one cell in direction d.

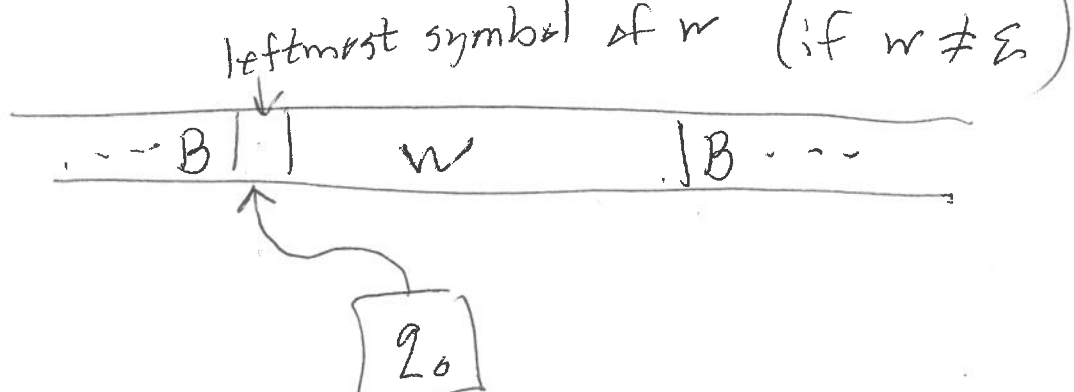
Transition diagram for a TM:

Generally

$$q \xrightarrow{a/b,\leftarrow} r \quad \text{means} \quad \delta(q,a) = (r,b,\leftarrow)$$

$$q \xrightarrow{a/b,\rightarrow} r \quad \delta(q,a) = (r,b,\rightarrow)$$

Initially, the input string $w \in \Sigma^*$ is on the tape, all other cells contain the blank symbol.

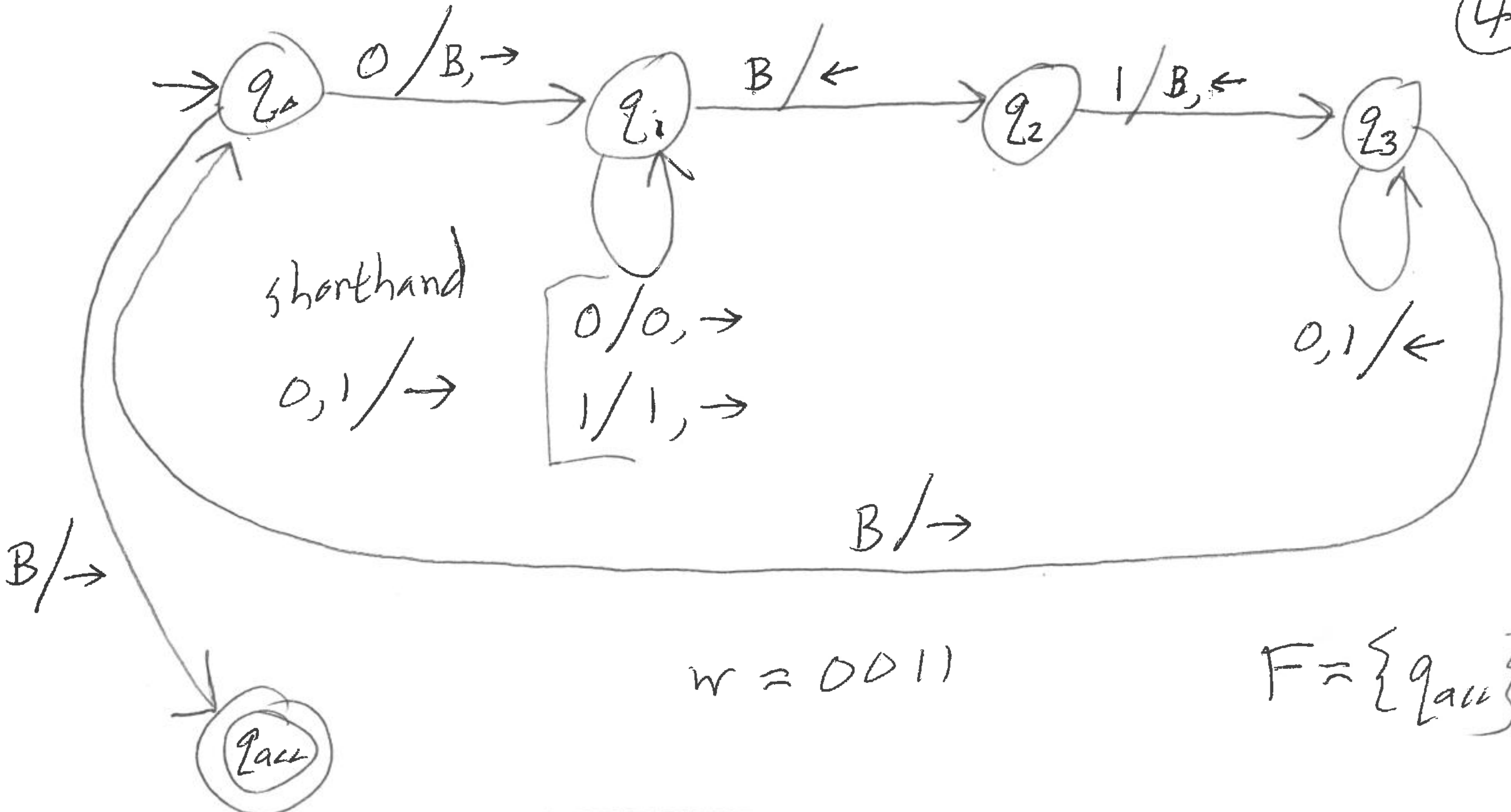


Computation ends when $\delta(q,a)$ is undefined.
Computation accepts here if $q \in F$ and rejects otherwise.

Ex: $L = \{0^n 1^n : n \geq 0\}$

$\Sigma = \{0,1\}$
$\Gamma = \{0,1,B\}$

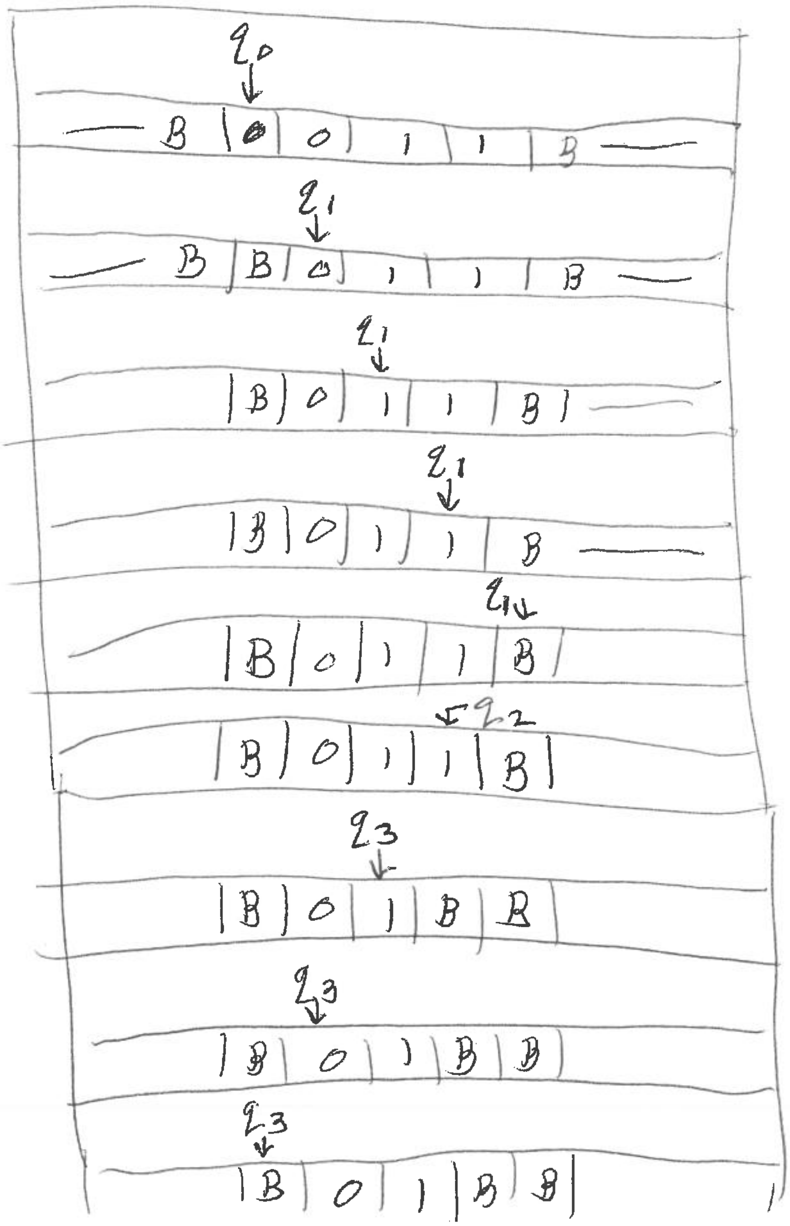
Transition diagram:



0/B,→
B/←
1/B,←
shorthand
0,1/→
0/0,→
1/1,→
0,1/←
B/→
B/→
q_0
q_1
q_2
q_3
q_{acc}

$w = 0011$

$F = \{q_{acc}\}$



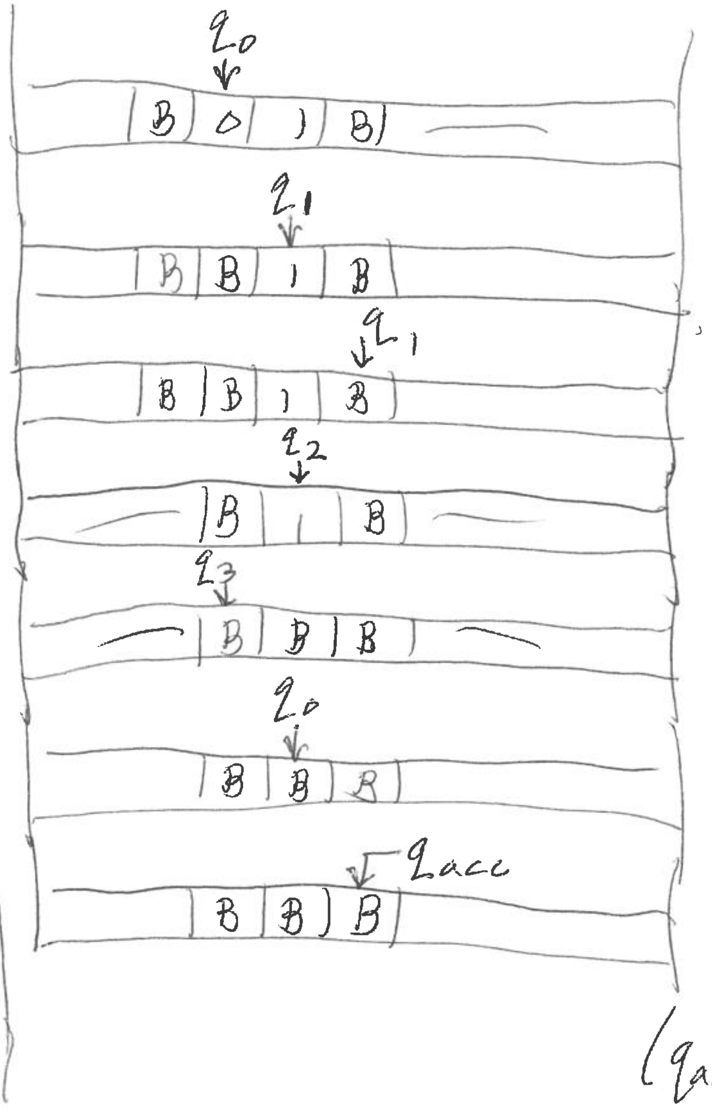
q_0
B | 0 | 0 | 1 | 1 | B
q_1
B | B | 0 | 1 | 1 | B
q_1
| B | 0 | 1 | 1 | B |
q_1
| B | 0 | 1 | 1 | B
q_1
| B | 0 | 1 | 1 | B |
q_2
| B | 0 | 1 | 1 | B |
q_3
| B | 0 | 1 | B | B |
q_3
| B | 0 | 1 | B | B |
q_3
| B | 0 | 1 | B | B |

Tabular form:

	0	1	B
→ q_0	$q_1, B, \rightarrow$	—	$q_{acc}, B, \rightarrow$
q_1	$q_1, 0, \rightarrow$	$q_1, 1, \rightarrow$	$q_2, B, \leftarrow$
q_2	—	$q_3, B, \leftarrow$	—
q_3	$q_3, 0, \leftarrow$	$q_3, 1, \leftarrow$	$q_0, B, \rightarrow$
* q_{acc}	—	—	—

— = undefined

(no arrow means undefined transition)



accept!

($q_{acc} \in F$)

Formal TM semantics

Def: Let $M = \langle Q, \Sigma, \Gamma, \delta, q_0, B, F \rangle$ be a TM. An instantaneous description (ID) or configuration is a string of the form $\alpha q \beta$ where

- $q \in Q$ (the "current state" of M)
- $\alpha, \beta \in \Gamma^*$ (the $\alpha\beta$ is "current nonblank contents of the tape") where

  $\alpha\beta$ is a string giving the a contiguous portion of the tape long enough to include
  - all nonblank symbols
  - the ~~symb~~ cell currently being scanned
- M is scanning the cell containing the first symbol of β.

Padding an ID on either side with blank symbols is allowed; we consider any padding to be the same ID.

Def: Let C be an ID of M:

~~of the~~ $C = \alpha q a \beta$ $(\alpha, \beta \in \Gamma^*, q \in Q, a \in \Gamma)$

The successor C' of C, if it exists is as follows:

— if $\delta(q, a) = (r, b, \rightarrow)$ for some $r \in Q$, $b \in \Gamma$

then $C' = \alpha b r \beta$

— if $\delta(q, a) = (r, b, \leftarrow)$

then $C' = \alpha' r c b \beta$ where $\alpha = \alpha' c$

c is last symbol of α

In either case, we say $C \vdash C'$.

On any input string $w \in \Sigma^*$, the initial config on input w is the string $q_0 w$

A computation on input w is a sequence of IDs

$C_0 \vdash C_1 \vdash C_2 \vdash \cdots$ where C_0 is the initial config on input w, and $C_i \vdash C_{i+1}$ for all i.