CSCE 355
4/7/2025

- Closure (& nonclosure) properties of the class of CFLs
- Turing Machines

①

Prop: CFLs are closed under $\cup$ (union), concat, $*$ operator (i.e. L_1, L_2 are CFLs, then so are $L_1 \cup L_2$, $L_1 L_2$, L_1^*)

Proof: Mimics the proof that every regular lang is a CFL (regex $\rightarrow$ CFG construction) //

Prop: CFLs closed under string reversal.

L is a CFL $\Rightarrow$ L^R is a CFL.

Proof sketch: Given a CFG G for L ($L = L(G)$)

Form G^R to be the same as G except if G has a production $A \rightarrow \alpha$, then G^R instead has $A \rightarrow \alpha^R$

Parse trees for G^R are left-to-right mirror images of those of G, and vice versa.

$\therefore$ yields are reversed, so $L^R = L(G^R)$. //

Recall $\varphi : \Sigma^* \to \Gamma^*$ is a string homomorphism when $\varphi(wx) = \varphi(w)\varphi(x)$ for all $w, x \in \Sigma^*$.

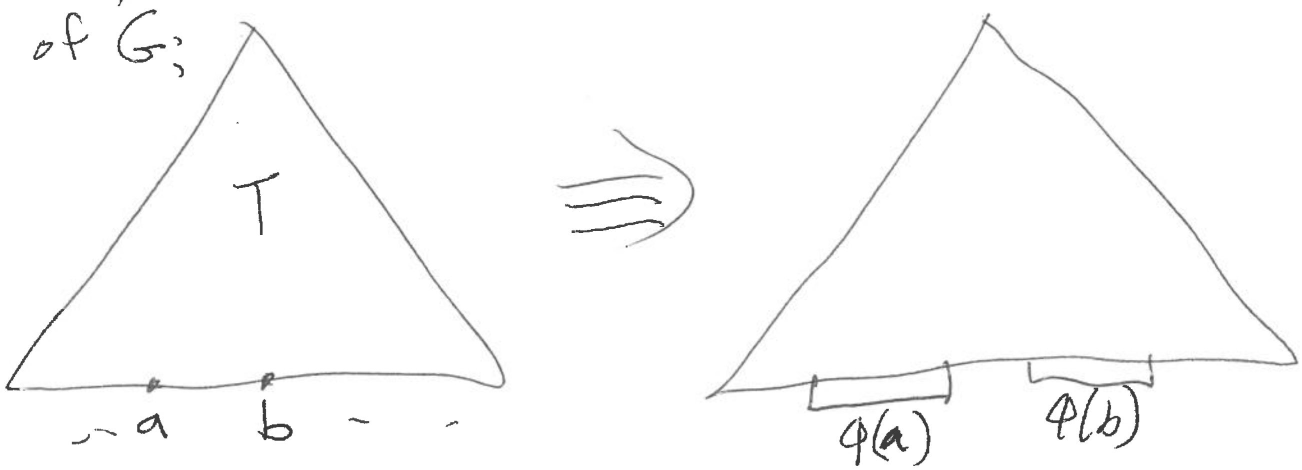
<u>Prop</u>: CFLs are closed under homom. images:

L is a CFL $\Longrightarrow$ $\varphi(L)$ is a CFL, for $\varphi : \Sigma^* \to \Gamma^*$ homomorphism.

<u>Proof idea</u>: Given a grammar G for L, replace each terminal $a \in \Sigma$ in productions with $\varphi(a)$.

Eg: $S \to aSb \mid \varepsilon \Longrightarrow S \to \varphi(a) S \varphi(b) \mid \underbrace{\varphi(\varepsilon)}_{\varepsilon}$

Any parse tree of G:



If G' is the resulting grammar, then $\varphi(L) = L(G')$. //

Recall: $\varphi^{-1}(L) = \{ w \in \Sigma^* \mid \varphi(w) \in L \}$ $\quad L \subseteq \Gamma^*$

<u>Prop</u>: If $\underset{\subseteq \Gamma^*}{L}$ is a CFL, then $\underbrace{\varphi^{-1}(L)}_{\subseteq \Sigma^*}$ is a CFL.

<u>Proof idea</u>: Given a PDA P such that $L = N(P)$, construct a PDA P' for $\varphi^{-1}(L)$:

For any $a \in \Sigma \cup \{\varepsilon\}$, P' reading a mimics what P would do reading $\varphi(a)$.

Then $\varphi^{-1}(L) = N(P')$. //

<u>Nonclosure properties</u>

<u>Prop</u>: CFLs are <u>not</u> closed under complements.

<u>Proof</u>: We define a language L that is a CFL but $\overline{L}$ is not a CFL:

not a CFL $\rightarrow$ $\overline{L} := \{ xx : x \in \{a,b\}^* \}$

CFL $\rightarrow$ $L := \{ w : w$ is not of the form xx for any $x \in \{a,b\}^* \}$

$\overline{L}$ is not CFL-pumpable:

Given $p > 0$, let $s := \underbrace{a^p b^p}_{x} \underbrace{a^p b^p}_{x} \in \overline{L}$.

Given any $u, v, w, x, y, \ldots$,

set $i := 0$.

L is CFL $\quad L = \{w : w \neq xx$ for any $x \in \Sigma^*\}$

$\Sigma = \{a, b\}$.

Grammar G for L:

$S \to AB \mid BA \mid O$ $\quad$ O — odd length

$C \to a \mid b$ $\quad$ C – single char

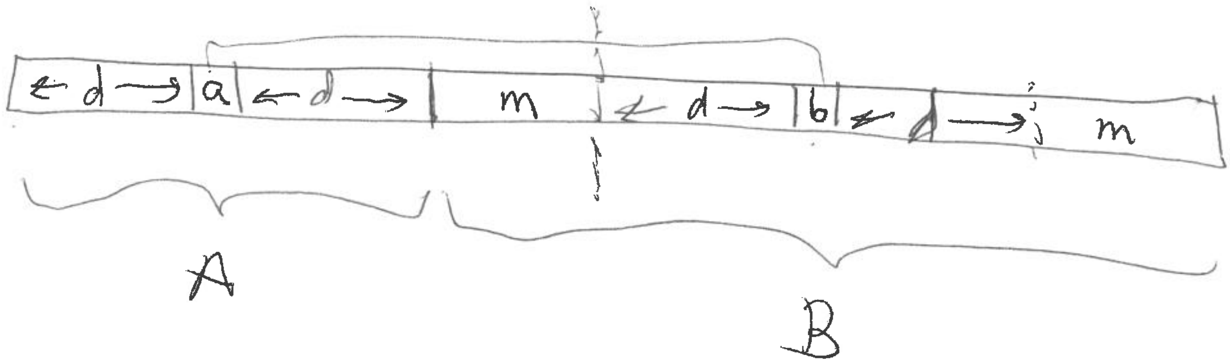
$A \to a \mid CAC$ $\quad$ A — any odd-length string with "a" in the middle

$B \to b \mid CBC$ $\quad$ B — any odd-length string with "b" in the middle

$O \to C \mid OCC$

Claim: $L = L(G)$.

Using $S \to AB \cdots$ yields



a, b are in the same positions within their respective halves, so yield is in L.

Similarly for $S \to BA \cdots$

Other chars in the string are arbitrary, so get all strings in L this way. //

<u>Prop</u>: CFLs are not closed under intersection.

<u>Proof</u>: We've proved most of this already.

$L_1 := \{a^m b^m c^n : m, n \geq 0\}$; $L_2 := \{a^m b^n c^n : m, n \geq 0\}$

$$\left.\begin{array}{l} S \to Sc \mid A \\ A \to aAb \mid \varepsilon \end{array}\right\} \text{for } L_1 \qquad \left.\begin{array}{l} S \to aS \mid B \\ B \to bBc \mid \varepsilon \end{array}\right\} L_2$$

But

$L_1 \cap L_2 = \{a^n b^n c^n : n \geq 0\}$.

Showed that $L_1 \cap L_2$ is not CFL-pumpable.

$\therefore L_1 \cap L_2$ not a CFL. //

Pumping Lemma for CFLs: Let L be any CFL. There exist $p>0$, $\forall s \in L$, $|s| \geq p$, $\exists u,v,w,x,y$ such that

$$s = uvwxy$$
$$|vwx| \leq p$$
$$|vx| > 0, \text{ and}$$

$\forall i \geq 0$, $uv^i wx^i y \in L$.

Proof: Let G be a CFG for L $(L = L(G))$.

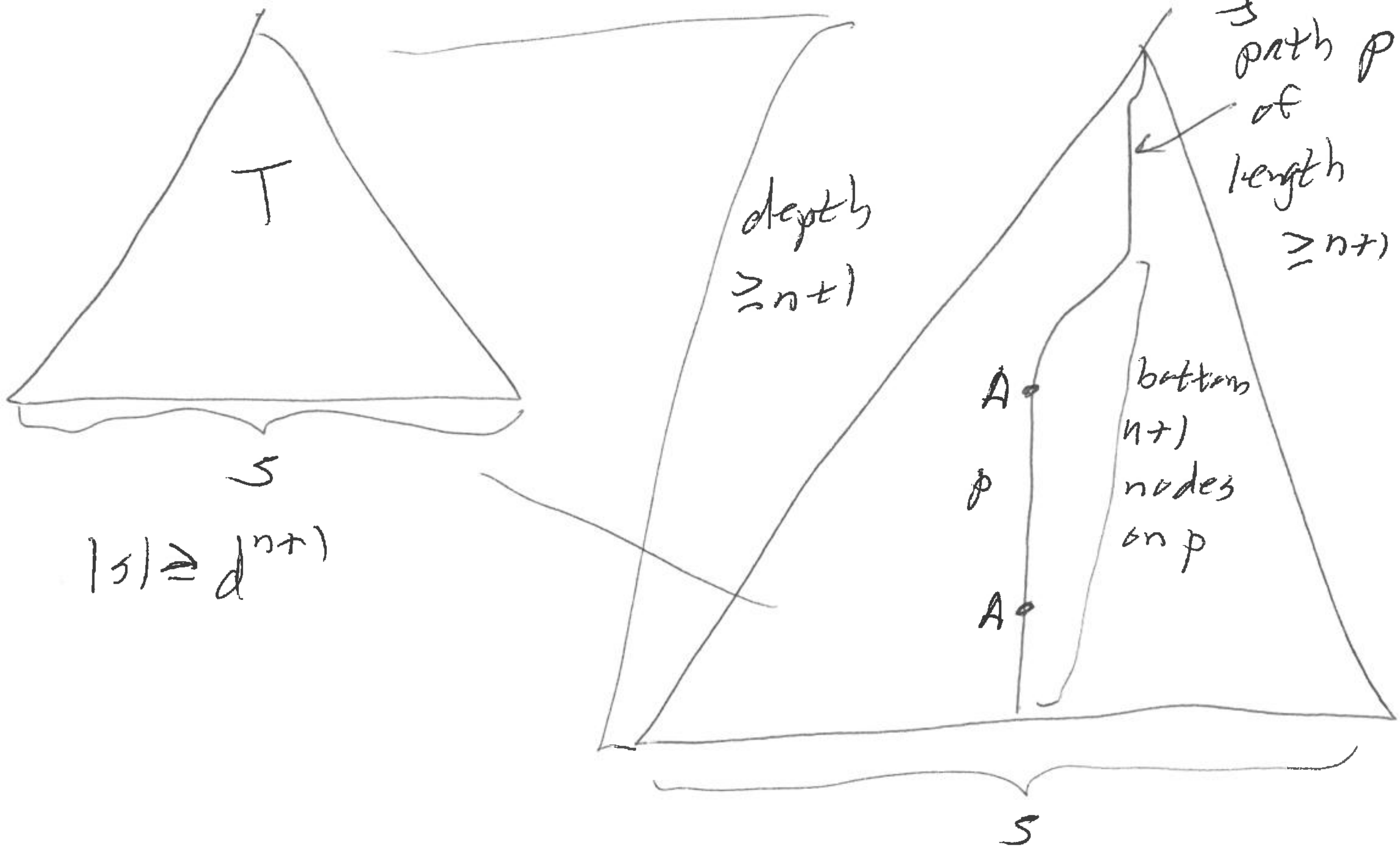
Let $d := \max\{|\alpha| : A \to \alpha \text{ is a production of } G, \text{ for some } A\}$

Let $n :=$ # of nonterminals of G.

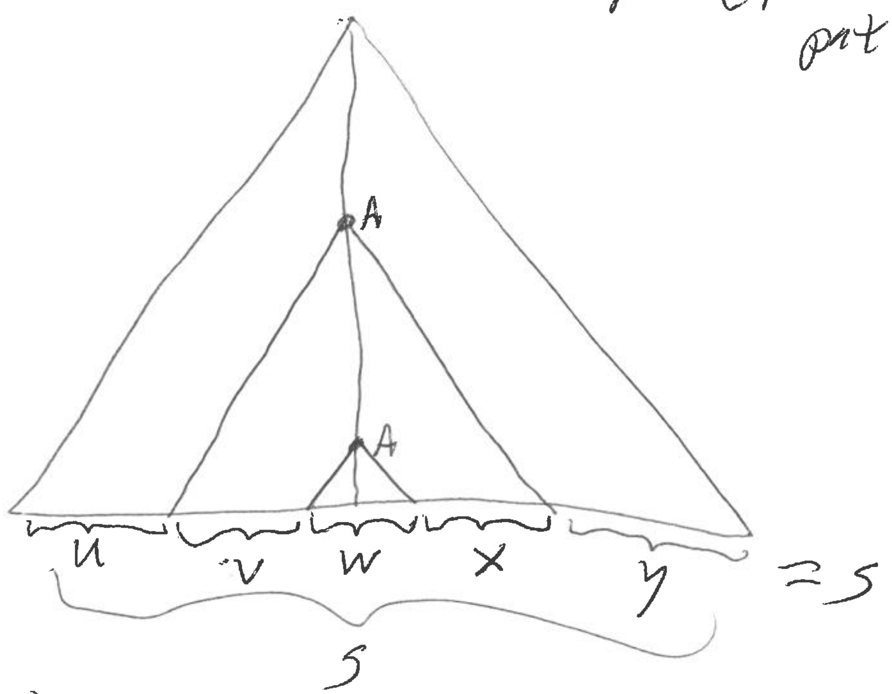
Set $\boxed{p := d^{n+1}}$.

Let $s \in L$ be arbitrary such that $|s| \geq \overset{d^{n+1}}{\overset{\shortparallel}{p}}$.

Let T be a parse tree of G of minimum size yielding s.



Some nonterminal A occurs $\geq$ twice among the bottom $n+1$ nodes in p. [p has max length path in T]

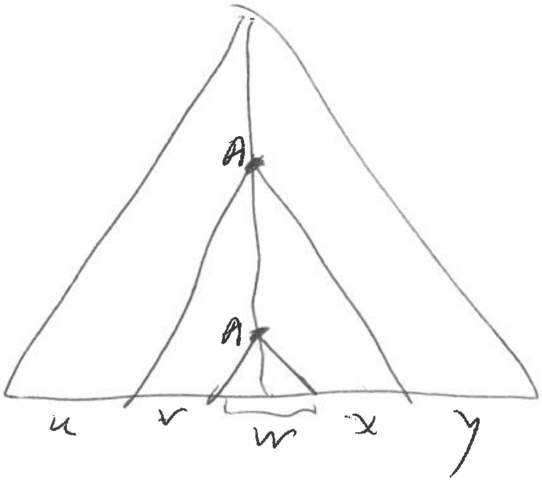


$|vwx| \leq d^{n+1} = p$ because upper A has height $\leq n+1$.

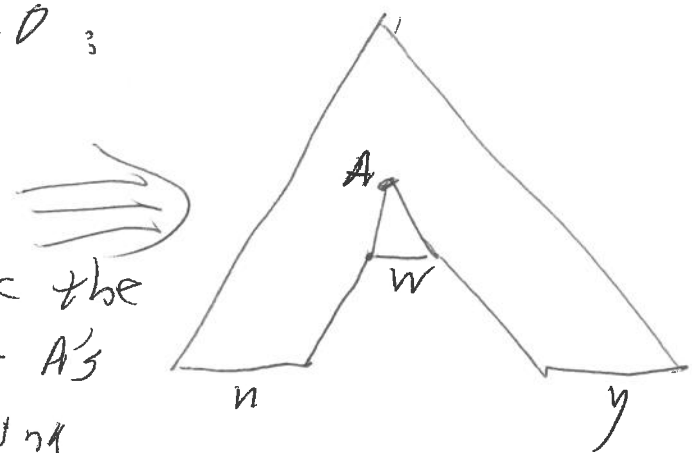
Pumping:

down:

$i := 0$;

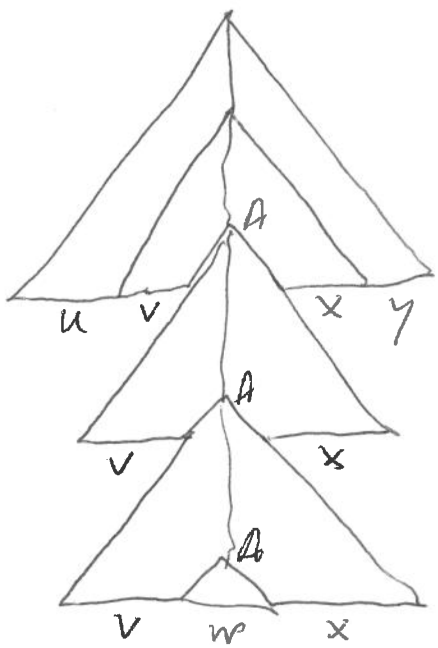


merge the two A's splicing out v and x



Pumping up:

$i := 3$;

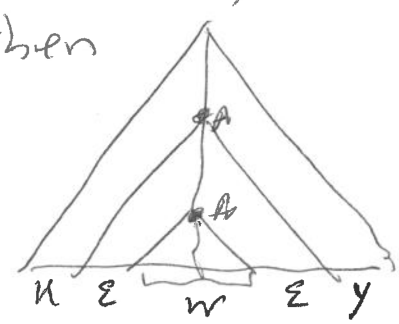


Parse tree for uv^3wx^3y

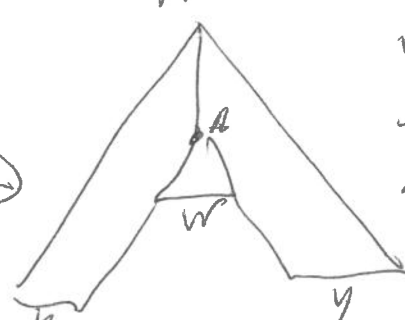
[works for any i]

~~Fina~~ Finally; claim that $|vx| > 0$. Suppose otherwise. $v = x =$

then



$\Rightarrow$ pump down: $\Rightarrow$



yielding the same string s

smaller tree yields s