CSCE 355
3/31/2025

$\text{PDA} \xrightarrow{\text{restricted PDA}} \text{CFG}$ ①

[Last time: CFG $\longrightarrow$ PDA]

Def: A restricted PDA is a PDA

$\langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle$

such that, if $\delta(q, a, X)$ contains (r, γ), then

$q \in Q$
$a \in \Sigma \cup \{\varepsilon\}$
$X \in \Gamma$

pop [either $\gamma = \varepsilon$ (X is popped)

push Y [or YX for some $Y \in \Gamma$ (Y is pushed)

Thm: For every PDA there is an equivalent restricted PDA [acceptance via empty stack]

Proof: Let $P := \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle$ be a PDA. We define a restricted PDA P' as follows:

P': bottom stack marker $X_0 \notin \Gamma$

$a \in \Sigma \cup \{\varepsilon\}$

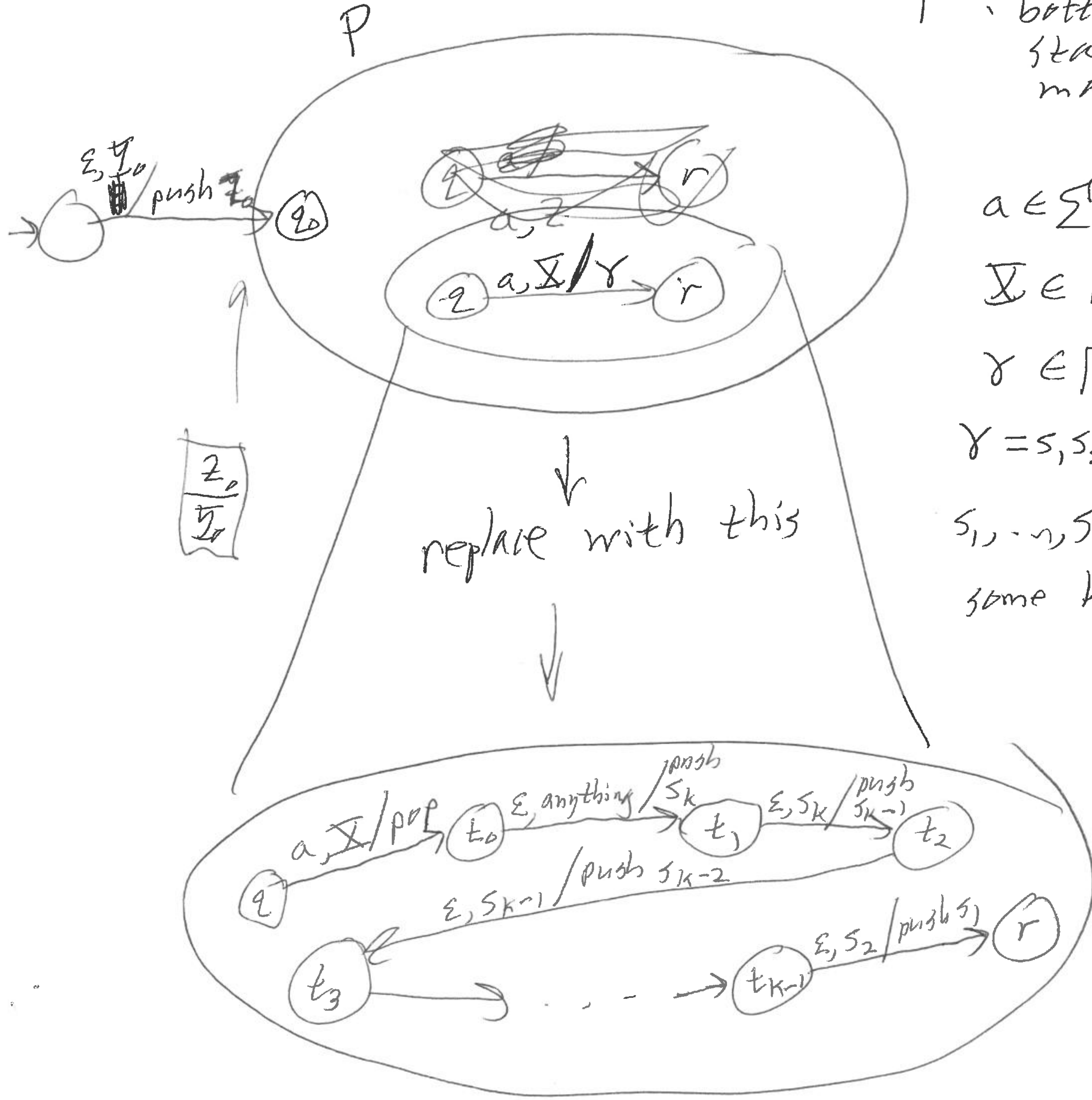
$X \in \Gamma$

$\gamma \in \Gamma^*$

$\gamma = s_1 s_2 \cdots s_k$

$s_1, \ldots, s_k \in \Gamma$

some $k \geq 0$



$t_0, \ldots, t_{k-1}$ are new states, different for each replacement.

Do this for every transition of P.

For every state $q \in Q$, add a transition

$q \xrightarrow{\varepsilon,\, X_0 / \text{pop}} q$

End of construction

Then $N(P') = N(P)$. Proof of correctness

<u>Thm</u>: For every restricted PDA P, there exists a CFG G such that $L(G) = N(P)$

<u>Proof</u>: By construction. Let $P := \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, \emptyset \rangle$ be a restricted PDA. Define a CFG G as follows: $G = \langle V, \Sigma, S, Prod \rangle$ where

$$V := \{S\} \cup \{[qXr] : q, r \in Q \text{ and } X \in \Gamma\}$$

disjoint union

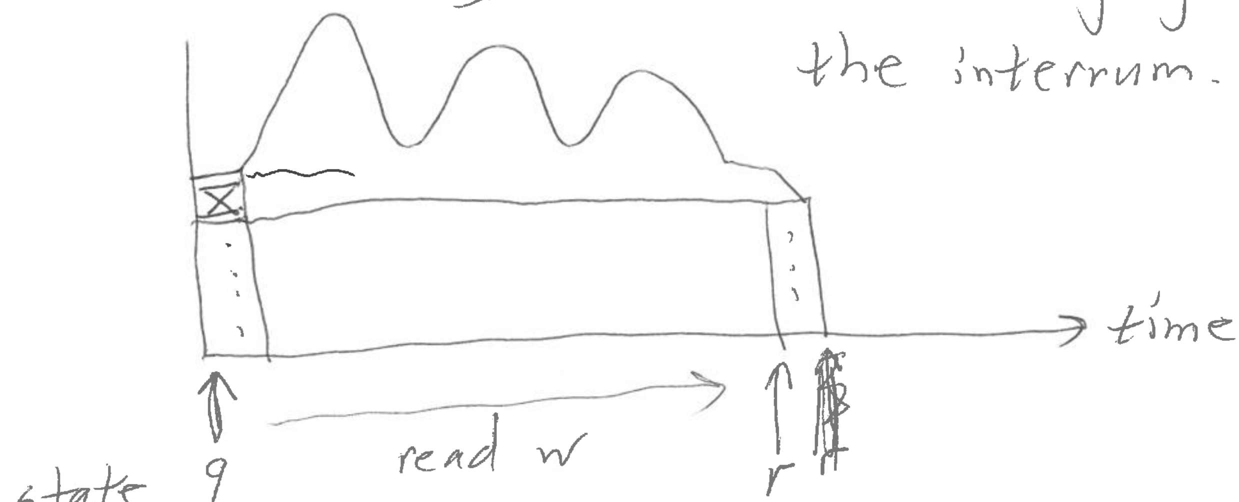
Productions in $Prod$:

$$S \to [q_0 Z_0 q_0] \mid [q_0 Z_0 q_1] \mid \cdots \mid [q_0 Z_0 q_n]$$

where $Q = \{q_0, q_1, \ldots, q_n\}$

<u>Idea</u>: $[qXr]$ derives all input strings w such that: P starting in state q ~~and~~ and ~~reading~~ X on top of the stack, can

read w and end in state r where the net change of the stack is that X is popped, with nothing underneath X on the stack changing in the interrum.



$$(q, w, X\gamma) \vdash \cdots \vdash (\ldots, \ldots, \ldots\gamma) \vdash \cdots \vdash (r, \varepsilon, \gamma)$$

Accepting computation of w looks like

$$(q_0, w, Z_0) \vdash \cdots \vdash (\ldots, \ldots, \ldots Z_0) \vdash \cdots \vdash (r, \varepsilon, \varepsilon)$$

Want ~~po~~ possible iff $S \Rightarrow^* w$

Recall: $S \to [q_0 Z_0 q_0] \mid \cdots \mid [q_0 Z_0 q_n]$

Add productions $S \to [q_0 Z_0 q]$ for all $q \in Q$

Productions of $[qXr]$:

If $\delta(q, a, X) \ni (s, \text{push } Y)$

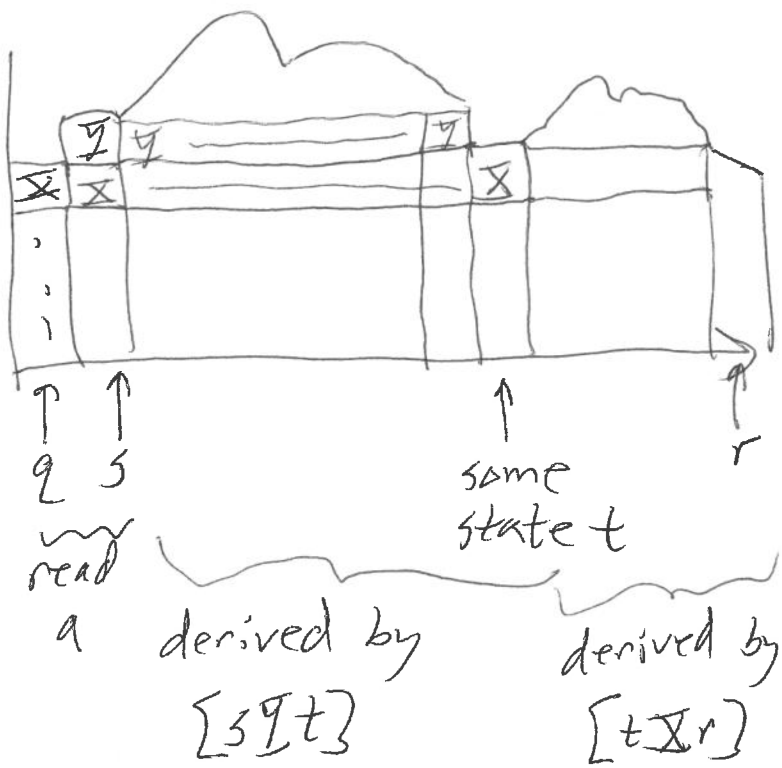
then add production

$\forall q, r \quad [qXr] \longrightarrow a[sYt][tXr]$ for every $t \in Q$

$\forall q, s$ If $\delta(q, a, X) \ni (s, \text{pop})$

then add production

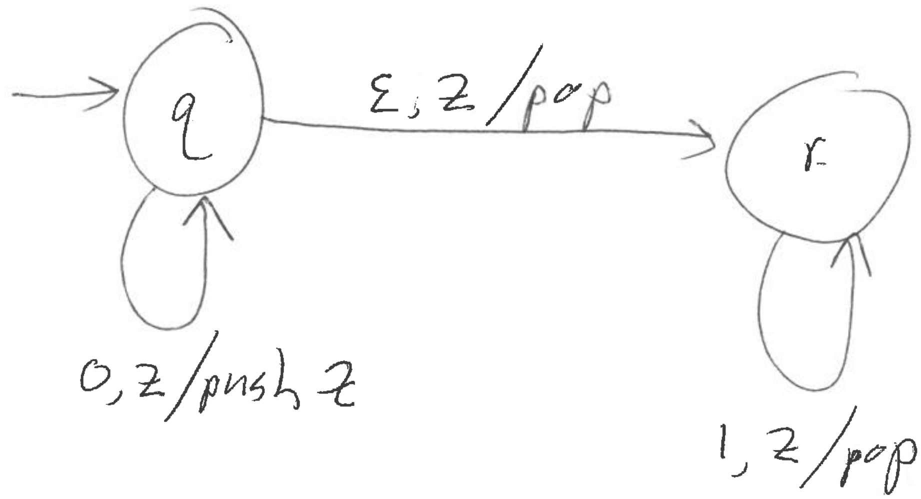
$$[qXs] \longrightarrow a$$



End of construction.

Example: restricted PDA $P = \langle \{q, r\}, \{0,1\}, \{Z\}, \delta, q, Z, \emptyset \rangle$

where $N(P) = \{ 0^n 1^n : n \geq 0 \}$



variables: S, $\underbrace{[qZq]}_{A}$, $\underbrace{[qZr]}_{B}$, $\underbrace{[rZq]}_{C}$, $\underbrace{[rZr]}_{D}$

$S \rightarrow A \mid B$

$[qZr] \rightarrow \varepsilon \qquad B \rightarrow \varepsilon$

$[rZr] \rightarrow 1 \qquad D \rightarrow 1$

$[qZq] \rightarrow 0[qZq][qZq] \mid 0[qZr][rZq]$

$[qZr] \rightarrow 0[qZq][qZr] \mid 0[qZr][rZr]$

Short hand:

$S \to A \mid B$

$B \to \varepsilon \mid 0AB \mid \cancel{0BD}$ $0B1$

$A \to 0AA \mid \overbrace{0BC}^{\text{not used}}$

no C productions

$\underbrace{D \to 1}_{\text{remove}}$

$S \to \underbrace{A}_{\text{remove}} \mid B$

$B \to \varepsilon \mid \overbrace{0AB}^{\text{remove}} \mid 0B1$

remove $\boxed{A} \to 0AA$

$S \to B$

$B \to \varepsilon \mid 0B1$

$S \to \varepsilon \mid 0S1$