CSCE 355
2/19/2025

Closure under inverse homomorphisms
Pumping lemma with examples

Recall: $\Sigma$, $\Gamma$ alphabets.

String homom. is a map $\varphi : \Sigma^* \to \Gamma^*$ such that $\forall x, y \in \Sigma^*$, $\varphi(xy) = \varphi(x)\varphi(y)$.

Completely determined by $\varphi$ on symbols from $\Sigma$.

$L \subseteq \Sigma^*$ language. $\varphi(L) := \{\varphi(w) : w \in L\} \subseteq \Gamma^*$.

$A \subseteq \Gamma^*$ language. $\varphi^{-1}(A) := \{w \in \Sigma^* : \varphi(w) \in A\} \subseteq \Sigma^*$.

Last time: If $L$ is regular, then $\varphi(L)$ is regular.
[Construction: given a regex, substitute $\varphi(a)$ for every symbol $a$ occurring in the regex.]

Now: If $A$ is regular, then $\varphi^{-1}(A)$ is regular.
Construction: given a DFA $D = \langle Q, \Gamma, \delta, q_0, F \rangle$ recognizing $A$. We define a DFA

$$D' = \langle Q, \Sigma, \delta', q_0, F \rangle,$$ where

$\forall q \in Q$ and $a \in \Sigma$, define

$$\delta'(q, a) := \hat{\delta}(q, \varphi(a))$$

so $q \xrightarrow{a} r$ in $D'$ means $q \xrightarrow{\varphi(a)} r$ in $D$

Claim is that $L(D') = \varphi^{-1}(A)$.



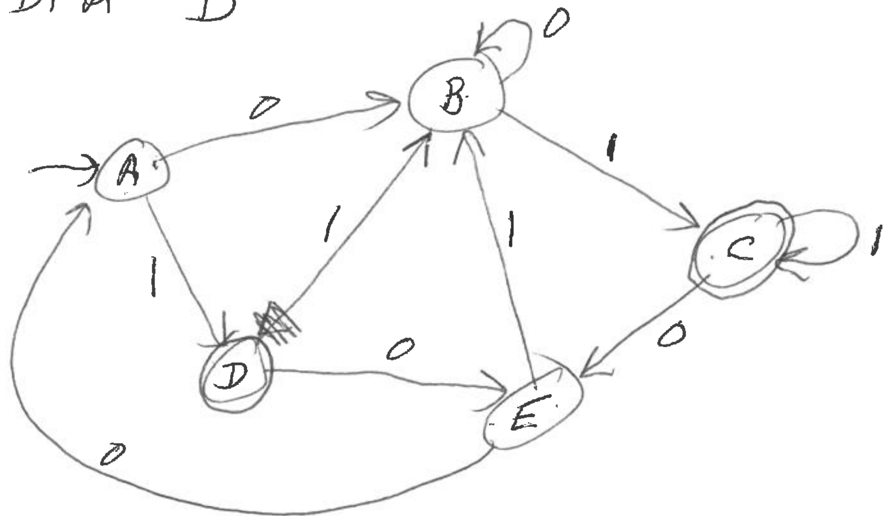
Ex: $\Sigma = \{a, b, c\}$, $\Gamma = \{0, 1\}$

$\varphi(a) = 011$

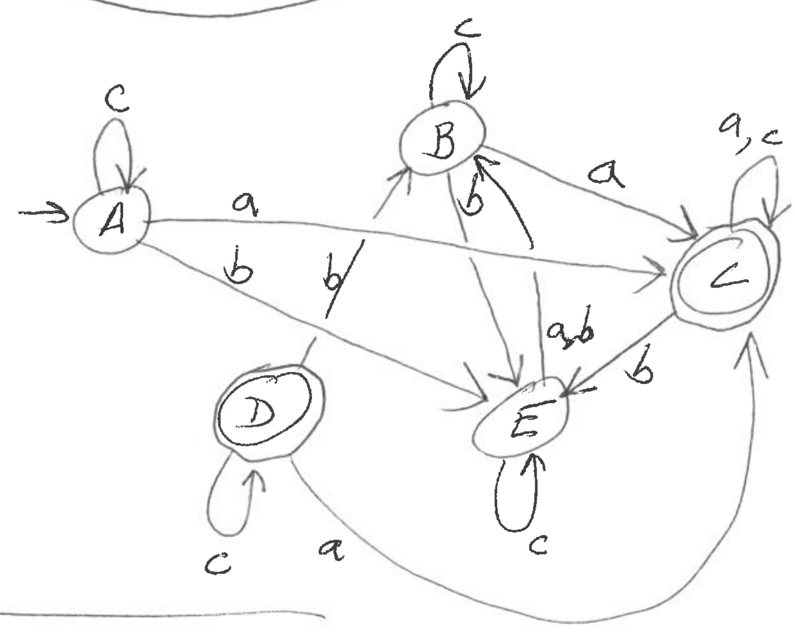
$\varphi(b) = 10$

$\varphi(c) = \varepsilon$

DFA D




D'




Proof of correctness:

Lemma: For any $w \in \Sigma^*$ and any $q \in Q$,

$$(\star) \quad \widehat{\delta'}(q, w) = \hat{\delta}(q, \varphi(w))$$

in D'     in D

Proof is by induction on $|w|$:

1) $w = \varepsilon$: $\hat{\delta}'(q, \varepsilon) = q = \hat{\delta}(q, \varepsilon) = \hat{\delta}(q, \varphi(\varepsilon))$

[showed last time that $\varphi(\varepsilon) = \varepsilon$]

2) $w = xa$ for some $x \in \Sigma'^*$ and $a \in \Sigma'$.

$$\hat{\delta}'(q, xa) = \delta'(\underline{\hat{\delta}'(q, x)}, a) \quad \left(\begin{array}{l}\text{ind def}\\ \text{of } \hat{\delta}'\end{array}\right)$$

$$= \delta'(\hat{\delta}(q, \varphi(x)), a)$$

[because $\hat{\delta}'(q, x) = \hat{\delta}(q, \varphi(x))$ by inductive hypothesis]

$$= \hat{\delta}(\hat{\delta}(q, \varphi(x)), \varphi(a))$$

(def of $\delta'$)

$\hat{\delta}(\hat{\delta}(q, x), y]$
$= \hat{\delta}(q, xy)$
$\forall x, y$ strings

$$= \hat{\delta}(q, \underline{\varphi(x)\varphi(a)})$$

$$= \hat{\delta}(q, \varphi(xa)) \quad [\varphi \text{ is a homom.}]$$

$\therefore$ by induction, ($\star$) holds for all $w$. // Lemma

Given any $w \in \Sigma'^*$, $w \in \varphi^{-1}(A)$

iff $\varphi(w) \in A$, iff $\underbrace{\hat{\delta}(q_0, \varphi(w)) \in F}_{D \text{ accepts } \varphi(w)}$

iff $\hat{\delta}'(q_0, w) \in F$ (by the lemma)

iff $D'$ accepts $w$.

$\therefore L(D') = \varphi^{-1}(A)$. $\therefore \varphi^{-1}(A)$ is regular. $\square$

$\Sigma = \{b, c\}$

Ex: ~~$\Gamma = \{a, b, c\}$. Given $L \subseteq \Sigma^*$, let $A(L)$ be the language obtained from L be taking any string in L~~

~~$\Sigma = \{b, c\}$, $\Gamma = \{a, b, c\}$.~~

$\Sigma = \{a, b, c\}$, $\Gamma = \{b, c\}$

Given $L \subseteq \Gamma^*$, let $A(L)$ be ~~obtained~~ the language over $\Sigma$ obtained by taking a string in $L$ and inserting any number of $a$'s anywhere in the string.

Ex: if $bbc \in L$, then $A(L)$ contains $bbc$, $abbc$, aabaaabacaaaa.



Prop: If $L$ is regular then $A(L)$ is regular

Proof idea: $A(L) = \varphi^{-1}(L)$ under the homomorphism $\varphi$ such that

$$\varphi(b) = b$$
$$\varphi(c) = c$$
$$\varphi(a) = \varepsilon$$

$w \in A(L) \iff \varphi(w) \in L$. //

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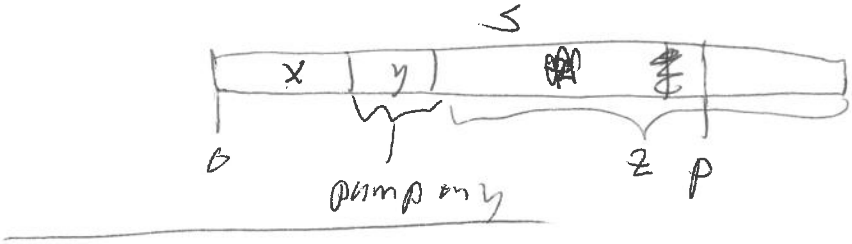
Pumping Lemma (for regular languages)

Def: Let $L$ be a language ($L \subseteq \Sigma^*$).
We say that $L$ is pumpable iff:

$p$ is the "pumping length"

There exists a $p > 0$ such that
For all $s \in L$ such that $|s| \geq p$,
There exist $x, y, z \in \Sigma^*$ such that
(1) $s = xyz$, (2) $|xy| \leq p$, (3) $y \neq \varepsilon$
and
for all $i \geq 0$, $xy^iz \in L$.
[$xz$, $xyz$, $xyyz$, $xyyyz$, ... ]




Pumping Lemma: Every regular language is pumpable

Proof: Spse $L$ is regular, so $L = L(D)$ for some DFA $D = \langle Q, \Sigma, \delta, q_0, F \rangle$

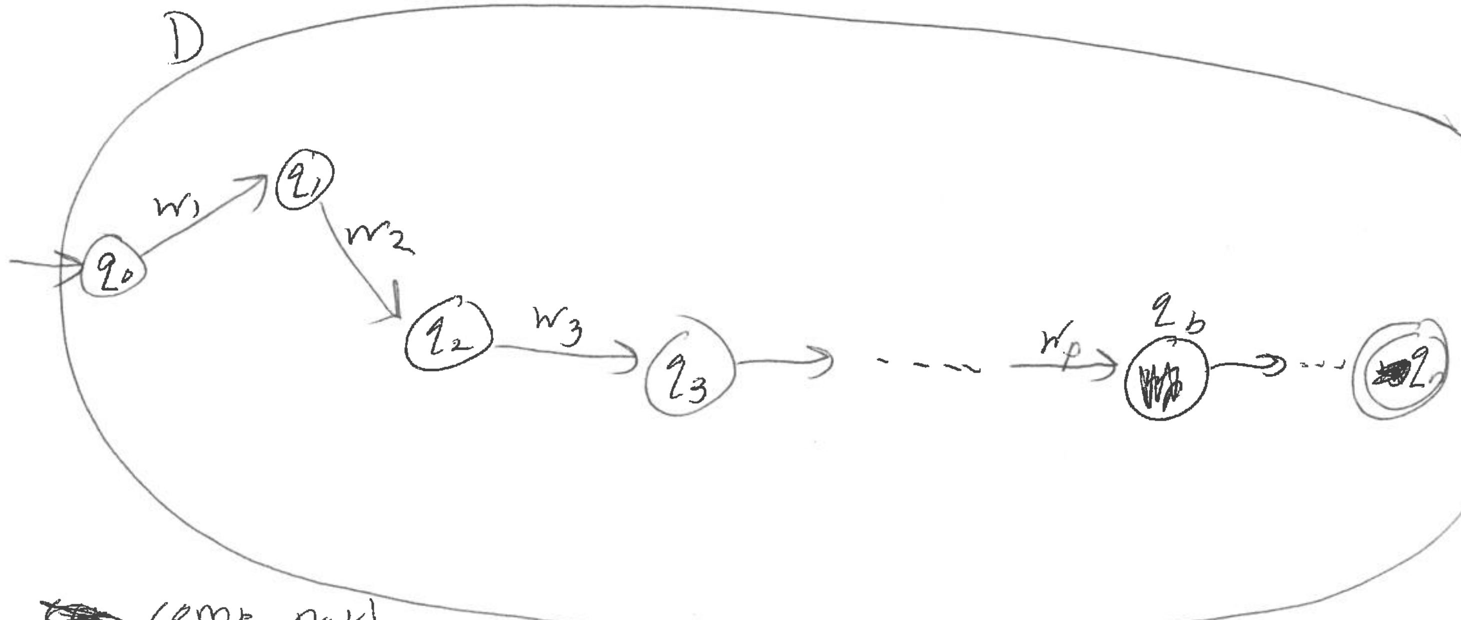
Let $p$ be the number of states of $D$

$(p := |Q|)$

Let $s \in L$ be given such that $|s| \geq p$

write $s = w_1 w_2 \cdots w_p \cdots w_n$ $(w_i \in \Sigma)$

$D$ accepts $s$,

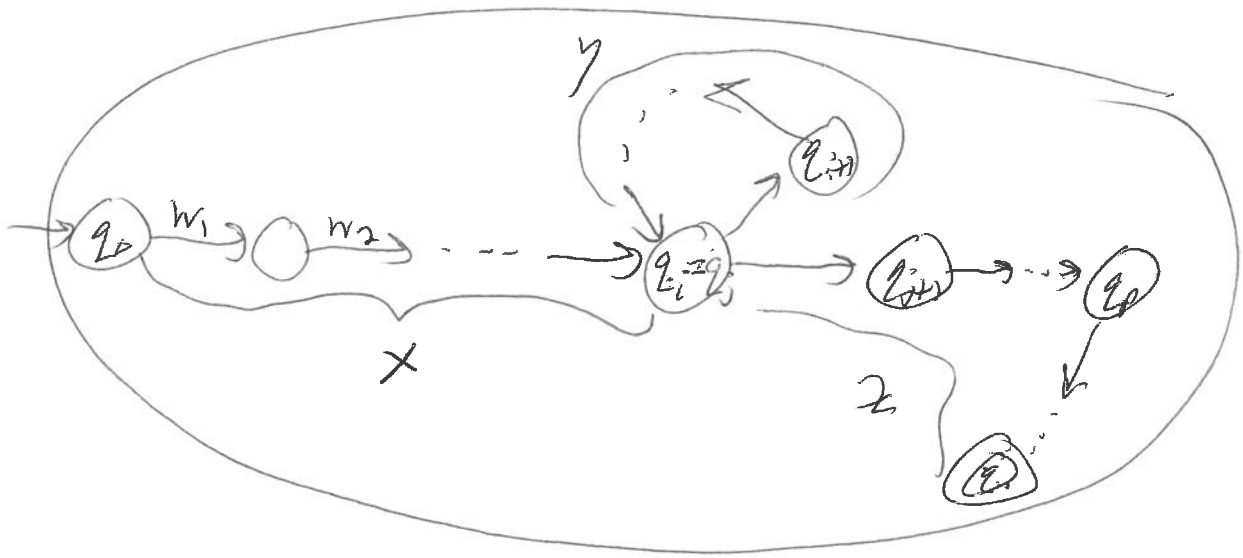



comp path

$\underbrace{q_0, q_1, \ldots, q_p}, \ldots, q_n$

$p+1$ elements in this segment,
but only $p$ states in $D$.

So $\exists\, i < j \leq p$, $q_i = q_j$ (Pigeonhole principle)




Let
$x := w_1 \cdots w_i$
$y := w_{i+1} \cdots w_j$
$z := w_{j+1} \cdots w_n$

Then $s = xyz$, $y \neq \varepsilon$, $|xy| = j \leq p$

Can read $xy^kz$ by taking the $q_i \rightsquigarrow q_j$ loop $k$ times, getting to the same accepting state as with $s$.

$\therefore\ xy^kz \in L \quad \forall k \geq 0$. ∎

<u>Prop</u>: Let $L := \{0^n 1^n : n \geq 0\}$.
L is not regular.

<u>Proof</u>: Suffices to show that L is not pumpable:

Given $p > 0$,

let $s := 0^p 1^p$. $(s \in L,\ |s| = 2p \geq p)$

Given $x, y, z$ such that

$s = xyz$, $|xy| \leq p$, $y \neq \varepsilon$,

(want $i \geq 0$ such that $xy^iz \notin L$)

[Know that $y = 0^k$ for some $k > 0$.]

let $i := 0$. ~~Then~~

Then $xy^iz = xz = 0^{p-k} 1^p \notin L$ because $p-k \neq p$

(removed $k$ many $0$'s from $s$)

Thus L is not pumpable, hence L is not regular by the Pumping Lemma //