CSCE 355
1/29/2025

Today: NFA $\to$ DFA

Theorem: For every NFA A, there is a DFA D such that $L(A) = L(D)$.

Proof: (by construction)

Def: For any set S, let $2^S = \{T : T \subseteq S\}$ (the powerset of S, also denoted $\mathcal{P}(S)$)

Given an NFA $A = \langle Q, \Sigma, \delta, q_0, F \rangle$

[recall: $\delta : Q \times \Sigma \to 2^Q$].

Construct a DFA $D := \langle 2^Q, \Sigma, \Delta, Q_0, \mathcal{F} \rangle$ where $Q_0 = \{q_0\}$

$$\mathcal{F} := \{ S \subseteq Q : S \cap F \neq \emptyset \}$$

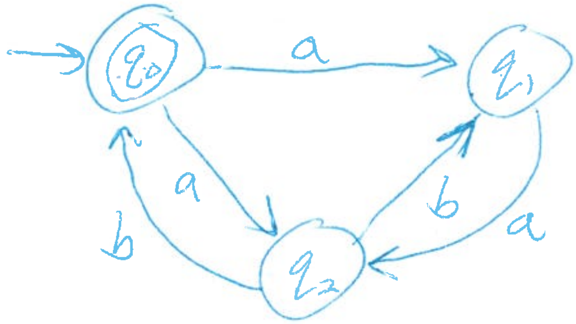
and for all $S \subseteq Q$ and $a \in \Sigma$, define

$$\Delta(S, a) := \bigcup_{q \in S} \delta(q, a).$$

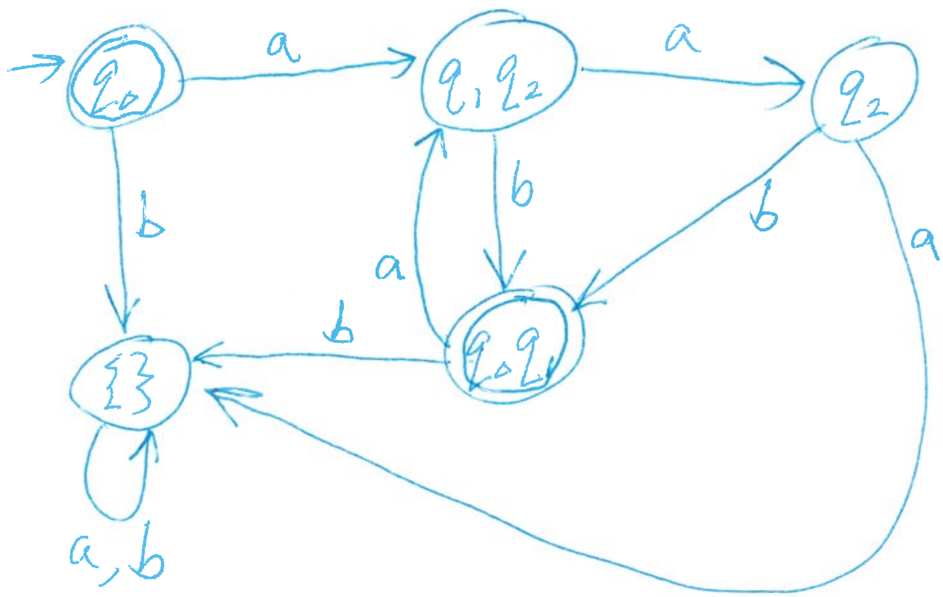
Then $L(D) = L(A)$ (proof by induction on input string length omitted). //

Ex: $\Sigma = \{a, b\}$

$A =$



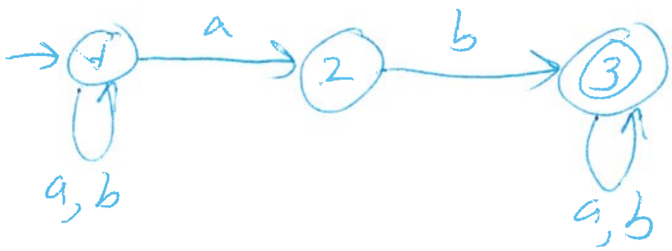
$D =$



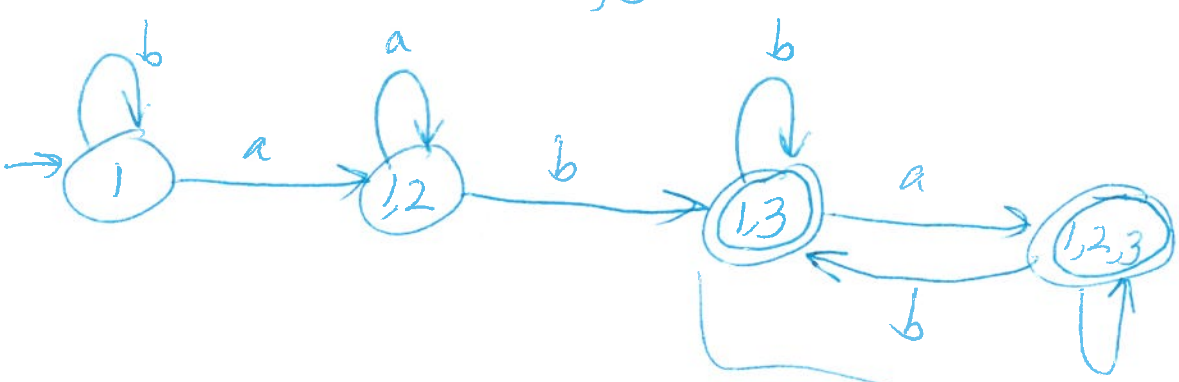
[or $\emptyset$]

Ex:

$A =$



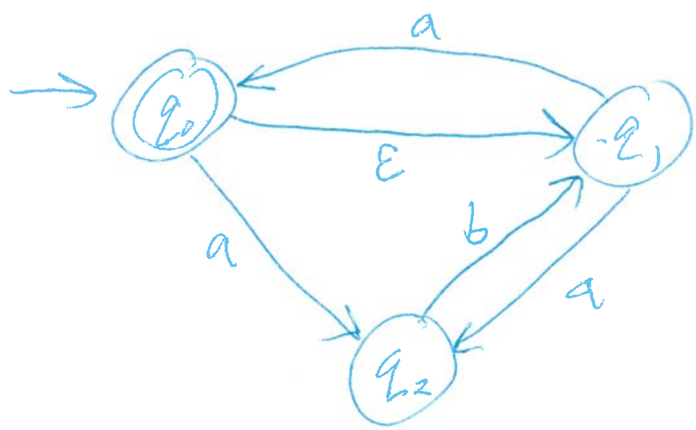
$D =$



can merge into one state

ε-Transitions:

ε-NFA



$q \xrightarrow{\varepsilon} r$ means can transition from q to r without consuming an input symbol.

Trivial: for every NFA there is an equivalent ε-NFA (same transition diagram).

Def: An ε-NFA is a tuple $\langle Q, \Sigma, \delta, q_0, F \rangle$ where Q, Σ, q_0, F are as with an NFA and

$$\delta : Q \times (\Sigma \cup \{\varepsilon\}) \longrightarrow 2^Q.$$

Def: Let $A = \langle Q, \Sigma, \delta, q_0, F \rangle$ be an ε-NFA, and let w be ~~an~~ a string over Σ.
A computation path of A on input w is a sequence of states $s_0, s_1, \ldots, s_n \in Q$

such that there exist $w_1, \dots, w_n \in \Sigma \cup \{\varepsilon\}$ such that

1.) $w = w_1 \cdots w_n$

2.) $s_0 = q_0$

3.) For all i, $1 \le i \le n$, $s_i \in \delta(s_{i-1}, w_i)$.

$$\left[\, q_0 = s_0 \xrightarrow{w_1} s_1 \xrightarrow{w_2} s_2 \xrightarrow{w_3} \cdots \xrightarrow{w_n} s_n \,\right]$$

s_n is the final state of the path

Say that A accepts w if there exists a computation path of A on input w whose final state is accepting (accepting path).

From the example: $w = abaa$ is accepted

accepting path: $q_0 \xrightarrow[w_1]{a} q_2 \xrightarrow[w_2]{b} q_1 \xrightarrow[w_3]{a} q_0 \xrightarrow[w_4]{\varepsilon} q_1 \xrightarrow[w_5]{a} q_0$

rejecting path: $q_0 \xrightarrow{a} q_2 \xrightarrow{b} q_1 \xrightarrow{a} q_0 \xrightarrow{a} q_2$

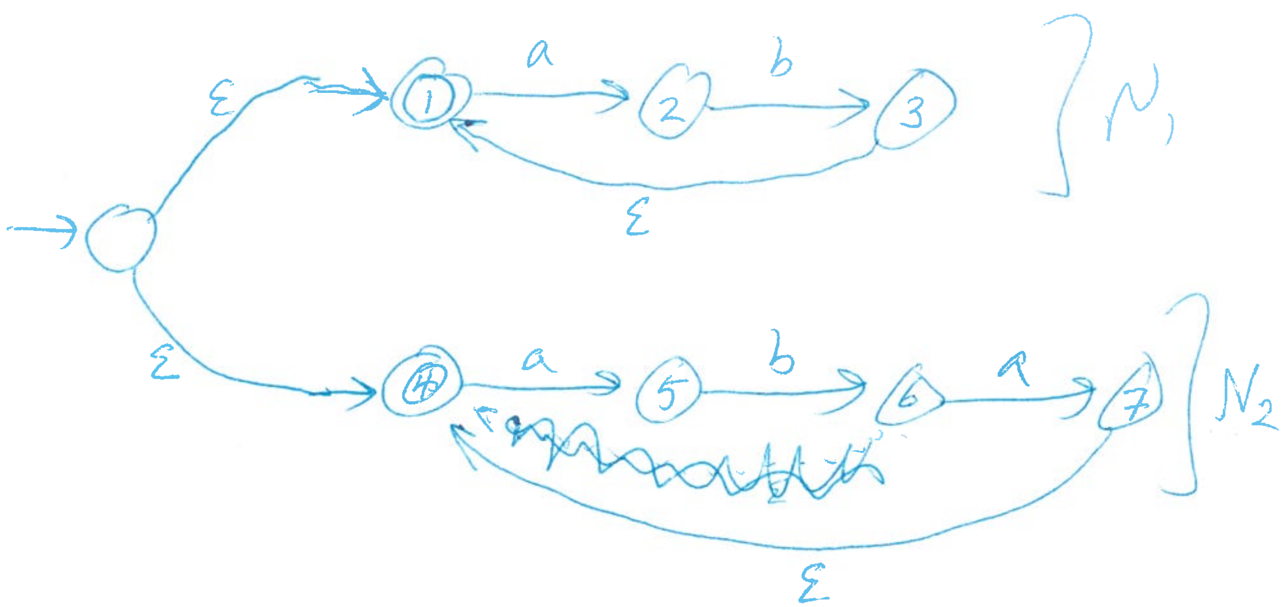
Tabular form for the example:

	a	b	ε
q_0	$\{q_2\}$	$\emptyset$	$\{q_1\}$
q_1	$\{q_0, q_2\}$	$\emptyset$	$\emptyset$
q_2	$\emptyset$	$\{q_1\}$	$\emptyset$

$\{q_0, q_2\}$

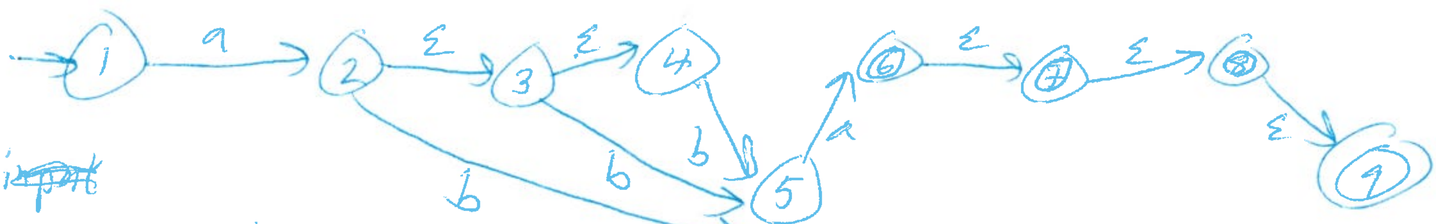
Ex: $L = \underbrace{\{(ab)^n : n \geq 0\}}\cup\underbrace{\{(aba)^n : n \geq 0\}}$ $\Sigma = \{a, b\}$

ε-NFA for L



Theorem: For every ε-NFA there is an equivalent NFA (with the same state set)

Proof Idea: will bypass ε-moves, then eliminate them.



~~input~~

accepting path on input aba

ε-NFA to NFA construction in 3 steps:

Step 1:

while there exist states $q, r \in Q$ such that $q \notin F$, $r \in F$, and $r \in \delta(q, \varepsilon)$ do

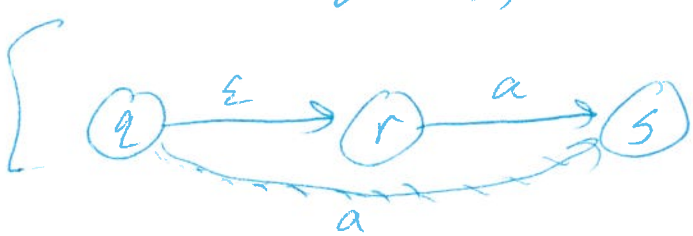
$F := F \cup \{q\}$ [make q accepting]

Step 2: **while** there exist states $q, r, s \in Q$ such that

$r \in \delta(q, \varepsilon)$,

$s \in \delta(r, a)$, (for some $a \in \Sigma$)

$s \notin \delta(q, a)$, do



$\delta(q, a) := \delta(q, a) \cup \{s\}$

Step 3: Remove all ε-transitions:

For all $q \in Q$, $\delta(q, \varepsilon) := \emptyset$.